Гармонический осциллятор в квантовой механике (продолжение)

Расчёт дисперсии координаты для произвольного состояния

$$n > 0$$

$$\langle n | (\hat{x} - \overline{x})^2 | n \rangle$$

$$\langle n | (\hat{x} - \overline{x})^2 | n \rangle = \langle n | \hat{x}^2 - 2\hat{x}\overline{x} + \overline{x}^2 | n \rangle =$$

$$= \langle n | \hat{x}^2 | n \rangle - 2\overline{x} \langle n | \hat{x} | n \rangle + \langle n | \overline{x}^2 | n \rangle =$$

$$\langle n | \hat{x} | n \rangle = \overline{x}$$

$$= \langle n | \hat{x}^2 | n \rangle - 2\overline{x} \cdot \overline{x} + \overline{x}^2 \langle n | n \rangle =$$

$$\langle n | n \rangle = 1$$

$$= \langle n | \hat{x}^2 | n \rangle - \overline{x} \cdot \overline{x}.$$

$\langle n | (\hat{x} - \overline{x})^2 | n \rangle = \langle n | \hat{x}^2 | n \rangle - \overline{x} \cdot \overline{x}$ – формула для дисперсии координаты

Расчёт $\langle n|\hat{x}|n\rangle$

$$\langle n|\hat{x}|n\rangle = ?$$

$$\hat{x} = \sqrt{\frac{\hbar}{m\omega_0}}\frac{1}{\sqrt{2}}\left(\hat{a}+\hat{a}^+\right)$$

$$\langle n|\hat{x}|n\rangle = \langle n|\sqrt{\frac{\hbar}{m\omega_0}}\frac{1}{\sqrt{2}}\left(\hat{a}+\hat{a}^+\right)|n\rangle =$$

$$= \langle n|\sqrt{\frac{\hbar}{m\omega_0}}\frac{1}{\sqrt{2}}\left(\hat{a}\right)|n\rangle + \langle n|\sqrt{\frac{\hbar}{m\omega_0}}\frac{1}{\sqrt{2}}\left(\hat{a}^+\right)|n\rangle =$$

Вспомним

$$\hat{a}|n\rangle = |n-1\rangle\sqrt{n}$$

$$\hat{a}^+|n\rangle = |n+1\rangle\sqrt{1+n},$$

$$= \sqrt{\frac{\hbar}{m\omega_0}}\frac{1}{\sqrt{2}}\langle n|n-1\rangle\sqrt{n} + \sqrt{\frac{\hbar}{m\omega_0}}\frac{1}{\sqrt{2}}\langle n|n+1\rangle\sqrt{1+n} = 0$$

$$\overline{x} = \langle n|x|n\rangle = 0$$

$\langle n|\left(\hat{x}-\overline{x}\right)^2|n\rangle = \langle n|\hat{x}^2|n\rangle$ − формула для дисперсии координаты

Расчёт $\langle n|\hat{x}^2|n\rangle$

$$\hat{x}=\sqrt{\frac{\hbar}{m\omega_0}}\frac{1}{\sqrt{2}}\left(\hat{a}+\hat{a}^+\right)$$

$$\hat{x}^2=\left(\sqrt{\frac{\hbar}{m\omega_0}}\frac{1}{\sqrt{2}}\right)^2\left(\hat{a}+\hat{a}^+\right)^2=\frac{\hbar}{2m\omega_0}\left(\hat{a}\hat{a}+\hat{a}\hat{a}^++\hat{a}^+\hat{a}+\hat{a}^+\hat{a}^+\right)$$

Вспомним

$$\hat{a}\hat{a}^+-\hat{a}^+\hat{a}=1\Rightarrow\hat{a}\hat{a}^+=1+\hat{a}^+\hat{a}$$

$$\hat{x}^2=\left(\sqrt{\frac{\hbar}{m\omega_0}}\frac{1}{\sqrt{2}}\right)^2\left(\hat{a}+\hat{a}^+\right)^2=\frac{\hbar}{2m\omega_0}\left(\hat{a}\hat{a}+2\hat{a}^+\hat{a}+1+\hat{a}^+\hat{a}^+\right)$$

$$\langle n|\hat{x}^2|n\rangle=$$

$$=\langle n|\frac{\hbar}{2m\omega_0}\left(\hat{a}\hat{a}+2\hat{a}^+\hat{a}+1+\hat{a}^+\hat{a}^+\right)|n\rangle=$$

$$=\frac{\hbar}{2m\omega_0}\left(\langle n|\hat{a}\hat{a}|n\rangle+\langle n|2\hat{a}^+\hat{a}|n\rangle\right)+$$

$$+\frac{\hbar}{2m\omega_0}\left(\langle n|n\rangle+\langle n|\hat{a}^+\hat{a}^+|n\rangle\right)=$$

$$=\frac{\hbar}{2m\omega_0}\left(\langle n|2\hat{a}^+\hat{a}|n\rangle\right)+$$

$$+\frac{\hbar}{2m\omega_0}\left(\langle n|n\rangle\right)=\frac{\hbar}{2m\omega_0}2n+\frac{\hbar}{2m\omega_0}=\frac{\hbar}{2m\omega_0}(2n+1)$$

$$\langle n|\hat{x}^2|n\rangle=\frac{\hbar}{2m\omega_0}(2n+1)$$

$$\langle n|\left(\hat{x}-\overline{x}\right)^2|n\rangle=\frac{\hbar}{2m\omega_0}(2n+1)\ \ \textbf{!!!}$$

Расчёт дисперсии импульса для произвольного состояния $n > 0$

$$\langle n|(\hat{p}-\overline{p})^2|n\rangle$$

$$\langle n|(\hat{p}-\overline{p})^2|n\rangle = \langle n|\hat{p}^2 - 2\overline{p}\hat{p} + \overline{p}^2|n\rangle =$$
$$= \langle n|\hat{p}^2|n\rangle - 2\overline{p}\langle n|\hat{p}|n\rangle + \langle n|\overline{p}^2|n\rangle.$$

$$\langle n|\overline{p}^2|n\rangle = \langle n|n\rangle\,\overline{p}^2 = \overline{p}^2$$

$$\langle n|\hat{p}|n\rangle = \overline{p}$$

$$\langle n|(\hat{p}-\overline{p})^2|n\rangle = \langle n|\hat{p}^2|n\rangle - 2\overline{p}\langle n|\hat{p}|n\rangle + \langle n|\overline{p}^2|n\rangle = \langle n|\hat{p}^2|n\rangle - \overline{p}^2$$

$$\underline{\langle n|(\hat{p}-\overline{p})^2|n\rangle = \langle n|\hat{p}^2|n\rangle - \overline{p}^2}$$

Расчёт $\langle n|\,\hat{p}\,|n\rangle$

$$\hat{p} = \sqrt{\hbar m\omega_0}\,\frac{1}{i\sqrt{2}}\left(\hat{a} - \hat{a}^+\right)$$

$$\langle n|\,\hat{p}\,|n\rangle = \langle n|\sqrt{\hbar m\omega_0}\,\frac{1}{i\sqrt{2}}\left(\hat{a} - \hat{a}^+\right)|n\rangle =$$

$$= \sqrt{\hbar m\omega_0}\,\frac{1}{i\sqrt{2}}\left(\langle n|\hat{a}|n\rangle - \langle n|\hat{a}^+\,|n\rangle\right) =$$

$$\hat{a}^+\,|n\rangle = |n+1\rangle\sqrt{1+n}\,,$$

$$\hat{a}\,|n\rangle = |n-1\rangle\sqrt{n}\,,$$

$$= \sqrt{\hbar m\omega_0}\,\frac{1}{i\sqrt{2}}\left(\langle n|n-1\rangle\sqrt{n} - \langle n|n=1\rangle\sqrt{1+n}\right) = 0$$

$$\langle n|\,\hat{p}\,|n\rangle = \overline{p} = 0$$

$$\langle n|\left(\hat{p} - \overline{p}\right)^2|n\rangle = \langle n|\,\hat{p}^2\,|n\rangle - \overline{p}^2$$

$$\underline{\langle n|\left(\hat{p} - \overline{p}\right)^2|n\rangle = \langle n|\,\hat{p}^2\,|n\rangle}$$

Расчёт $\langle n|\,\hat{p}^2\,|n\rangle$

$$\hat{p}=\sqrt{\hbar m\omega_0}\,\frac{1}{i\sqrt{2}}\left(\hat{a}-\hat{a}^{+}\right)$$

$$\langle n|\,\hat{p}^2\,|n\rangle=\left(\sqrt{\hbar m\omega_0}\,\frac{1}{i\sqrt{2}}\right)^2\langle n|\left(\hat{a}-\hat{a}^{+}\right)^2|n\rangle=$$

$$=\left(\sqrt{\hbar m\omega_0}\,\frac{1}{i\sqrt{2}}\right)^2\langle n|\left(\hat{a}\hat{a}-\hat{a}\hat{a}^{+}-\hat{a}^{+}\hat{a}+\hat{a}^{+}\hat{a}^{+}\right)|n\rangle=$$

$$\hat{a}\hat{a}^{+}-\hat{a}^{+}\hat{a}=1\Rightarrow\hat{a}\hat{a}^{+}=1+\hat{a}^{+}\hat{a}$$

$$=-\frac{\hbar m\omega_0}{2}\langle n|\left(\hat{a}\hat{a}-1-2\hat{a}^{+}\hat{a}+\hat{a}^{+}\hat{a}^{+}\right)|n\rangle=$$

$$=-\frac{\hbar m\omega_0}{2}\langle n|\,\hat{a}\hat{a}\,|n\rangle+\frac{\hbar m\omega_0}{2}\langle n|n\rangle-$$

$$+\frac{\hbar m\omega_0}{2}\langle n|\,2\hat{a}^{+}\hat{a}\,|n\rangle-\frac{\hbar m\omega_0}{2}\langle n|\,\hat{a}^{+}\hat{a}^{+}\,|n\rangle=\frac{\hbar m\omega_0}{2}\left(2n+1\right)$$

$$\langle n=0|\,\hat{p}^2\,|n=0\rangle=\frac{\hbar m\omega_0}{2}\left(2n+1\right)$$

$$\langle n|\left(\hat{p}-\overline{p}\right)^2|n\rangle=\frac{\hbar m\omega_0}{2}\left(2n+1\right)\quad\textbf{!!!}$$

ИТОГ

$$\langle n|\left(\hat{x}-\overline{x}\right)^2|n\rangle=\frac{\hbar}{2m\omega_0}\left(2n+1\right)\quad\textbf{!!!}$$

$$\langle n|\left(\hat{p}-\overline{p}\right)^2|n\rangle=\frac{\hbar m\omega_0}{2}\left(2n+1\right)\quad\textbf{!!!}$$

$$\underline{\langle n|\left(\hat{x}-\overline{x}\right)^2|n\rangle\langle n|\left(\hat{p}-\overline{p}\right)^2|n\rangle=\frac{\hbar^2}{4}\left(2n+1\right)^2}$$

Расчёт волновой функции 1 возбуждённого состоянии

Вспомним

$$\psi_0\left(x\right)=\left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega_0}{2\hbar}x^2\right)$$ – волновая функция основного состояния,

$$\hat{a}^+\left|n\right\rangle=\left|n+1\right\rangle\sqrt{1+n}\ ,$$

Если $n=0$, то $\hat{a}^+\psi_0\left(x\right)=\psi_1\left(x\right)$

$$\hat{a}^+=\left(\hat{Q}-i\hat{P}\right)\frac{1}{\sqrt{2}}$$

$$\hat{Q}=\hat{x}\sqrt{\frac{m\omega_0}{\hbar}},\ \ \hat{P}=\hat{p}\frac{1}{\sqrt{\hbar m\omega_0}}$$

$$\hat{x}=x,\ \ \hat{p}=\frac{\hbar}{i}\frac{\partial}{\partial x}$$

$$\hat{a}^+=\frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega_0}{\hbar}}x-\hbar\frac{1}{\sqrt{\hbar m\omega_0}}\frac{\partial}{\partial x}\right)$$

$$\psi_1\left(x\right)=\left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4}\sqrt{2}\sqrt{\frac{m\omega_0}{\hbar}}x\exp\left(-\frac{m\omega_0}{2\hbar}x^2\right)$$

$$\psi_0(x) = \frac{1}{\sqrt{x_0\sqrt{\pi}}}\exp\left(-\frac{1}{2}\left(\frac{x}{x_0}\right)^2\right)$$

$$\psi_1(x) = \frac{1}{\sqrt{2x_0\sqrt{\pi}}}\left(2\left(\frac{x}{x_0}\right)\right)\exp\left(-\frac{1}{2}\left(\frac{x}{x_0}\right)^2\right), \ x_0 = \sqrt{\frac{\hbar}{m\omega_0}}$$

Волновая функция нестационарного состояния

$$\boldsymbol{\psi}(x,t) = c_1\psi_0(x)e^{-i\frac{E_0 t}{\hbar}} + c_2\psi_1(x)e^{-i\frac{E_1 t}{\hbar}}, \ E_0 = \frac{\hbar\omega_0}{2}, \ E_1 = \frac{3\hbar\omega_0}{2}$$

$$c_1 = c_2 = \frac{1}{\sqrt{2}}$$

$$\boldsymbol{\psi}(x,t) = \frac{1}{\sqrt{2}}\left(\psi_0(x)e^{-i\frac{E_0 t}{\hbar}} + \psi_1(x)e^{-i\frac{E_1 t}{\hbar}}\right)$$

$$\left|\boldsymbol{\psi}(x,t)\right|^2 = \frac{1}{2}\left(\psi_0^2(x) + \psi_1^2(x) + \psi_0(x)\psi_1(x)\left(e^{-i\frac{(E_0-E_1)t}{\hbar}} + e^{i\frac{(E_0-E_1)t}{\hbar}}\right)\right)$$

$$E_1 - E_0 = \hbar\omega_0$$

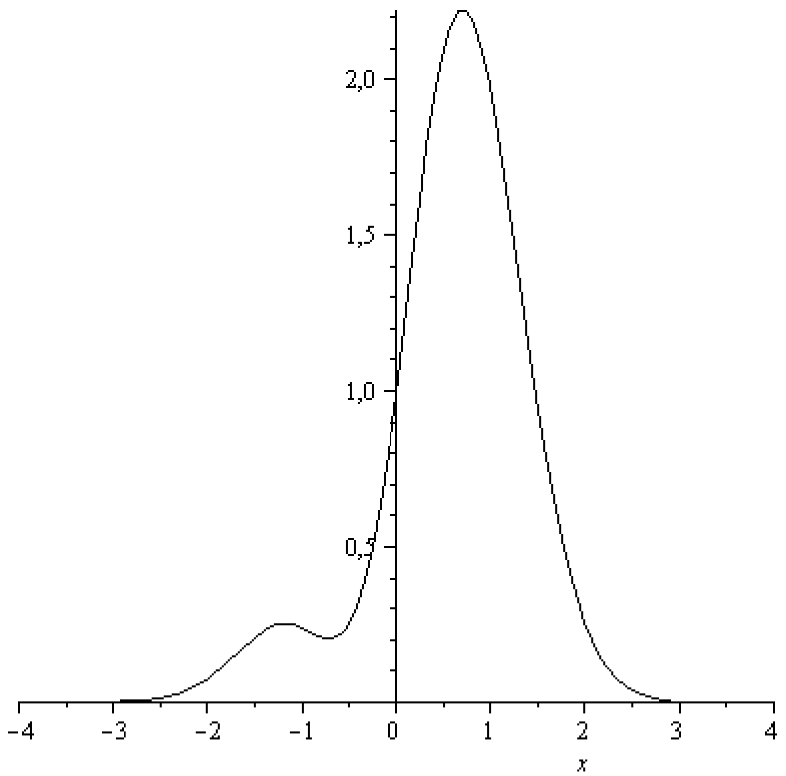
$$\left|\boldsymbol{\psi}(x,t)\right|^2 = \frac{1}{2}\left(\psi_0^2(x) + \psi_1^2(x) + 2\psi_0(x)\psi_1(x)\cos(\omega_0 t)\right)$$

$$\omega_0 t_n = \tau_n$$

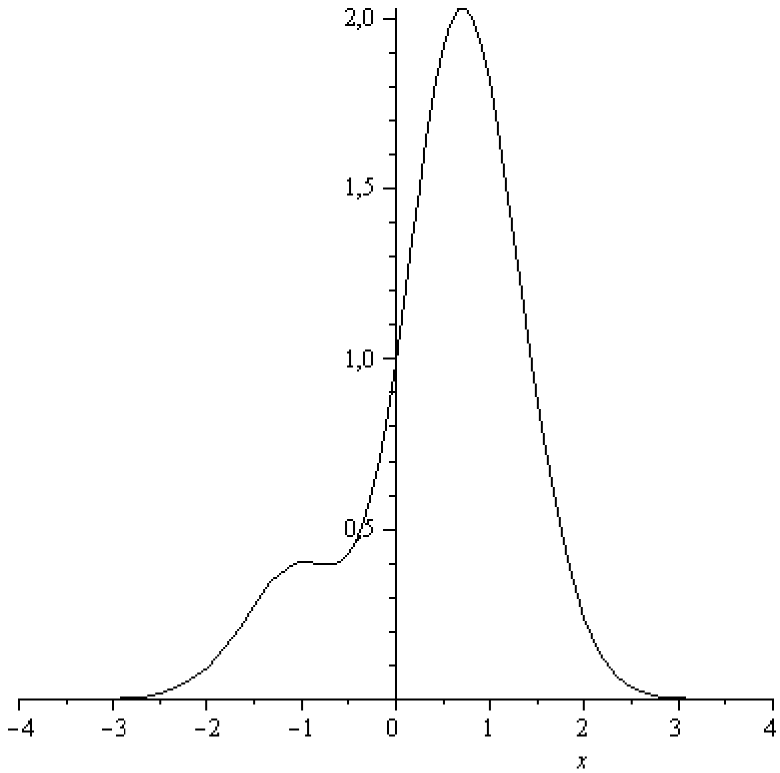
$$\tau_n = 2\pi\cdot 0; 2\pi\cdot\frac{1}{10}; 2\pi\cdot\frac{2}{10}; 2\pi\cdot\frac{3}{10}; ...2\pi\cdot\frac{10}{10};$$

$$\frac{x}{x_0} = x'$$

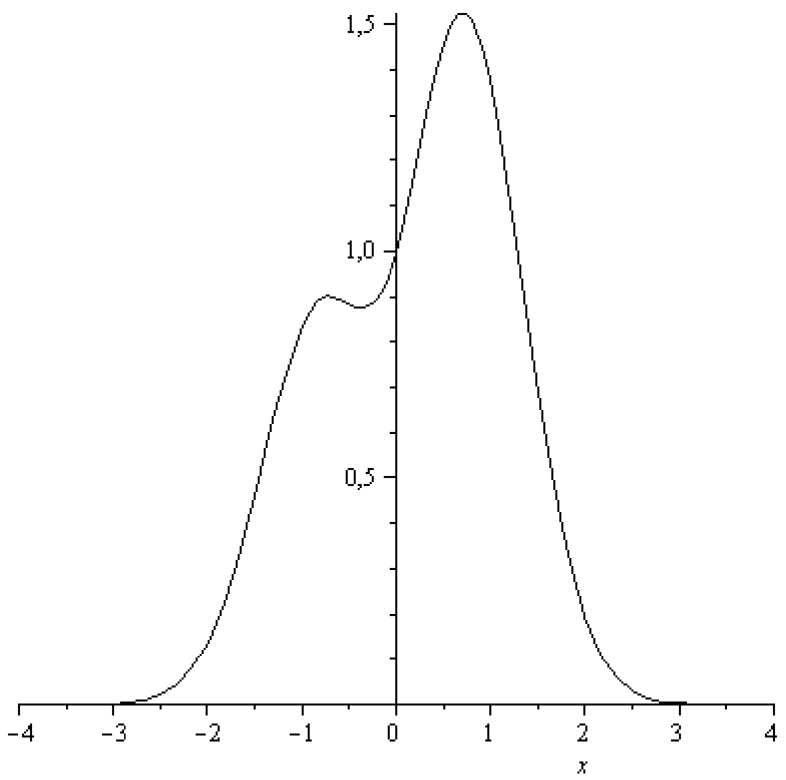
$$2x_0\sqrt{\pi}\left|\boldsymbol{\psi}(x,t)\right|^2 = \left(1 + 2(x')^2 + \frac{4}{\sqrt{2}}(x')\cos(\omega_0 t)\right)\exp\left(-(x')^2\right)$$



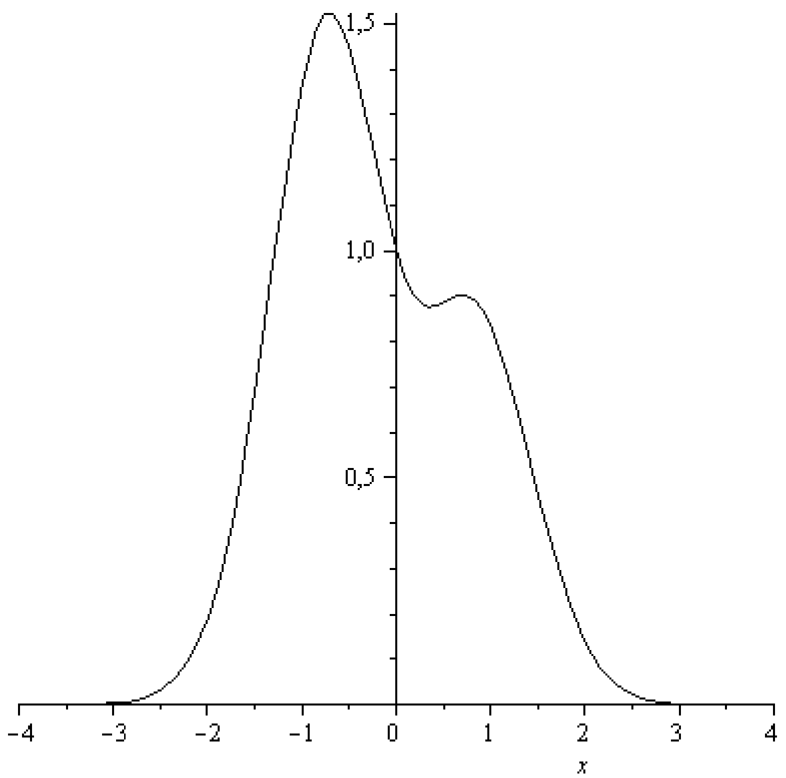
Зависимость $2x_0\sqrt{\pi}\left|\boldsymbol{\psi}\left(x,t_n\right)\right|^2$ $\tau_n = 0$



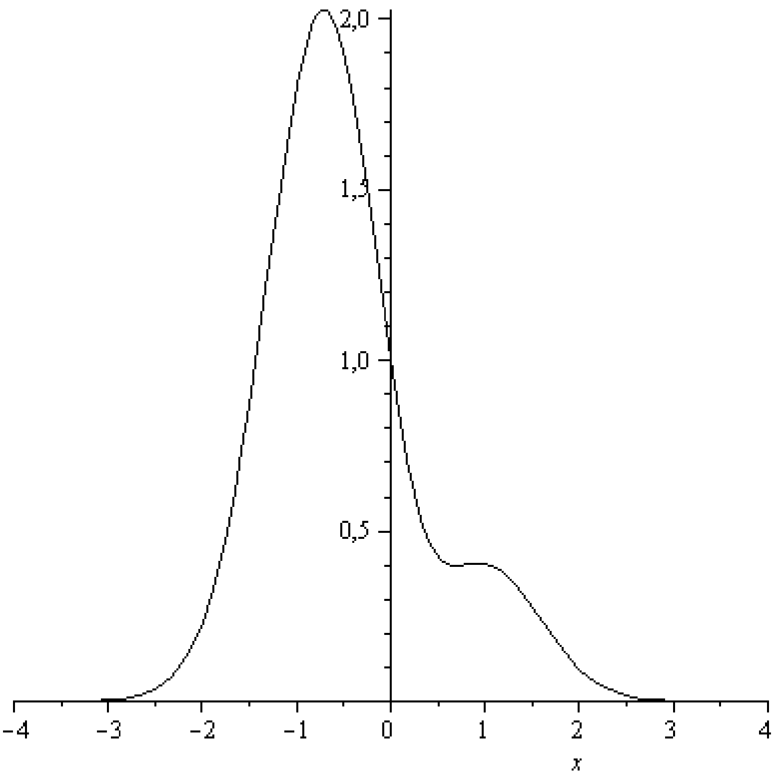
Зависимость $2x_0\sqrt{\pi}\left|\psi\left(x,t_n\right)\right|^2 \quad \tau_n = 0.1$



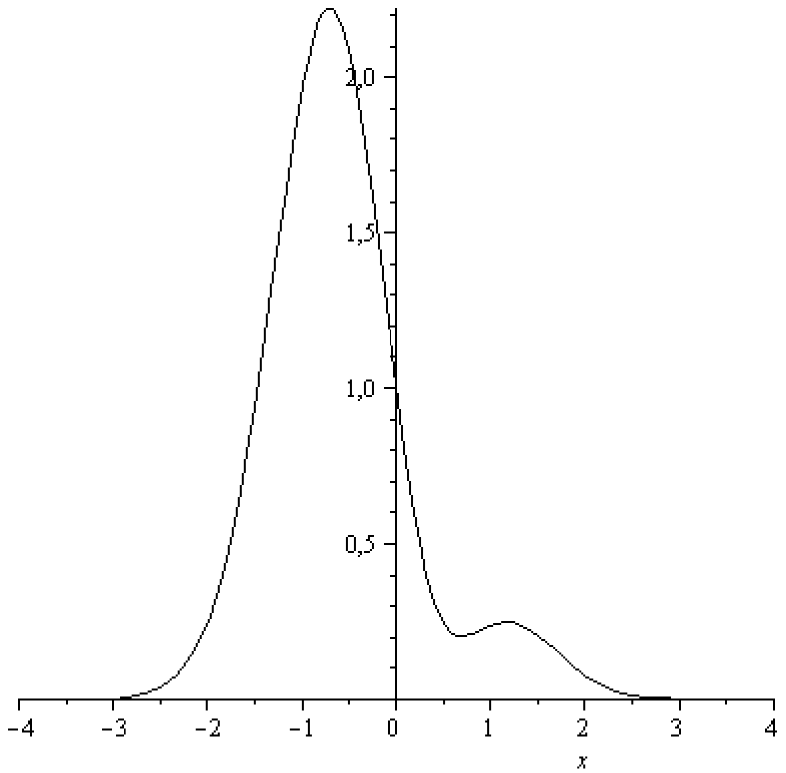
Зависимость $2x_0\sqrt{\pi}\left|\psi\left(x,t_n\right)\right|^2 \quad \tau_n = 0.2$



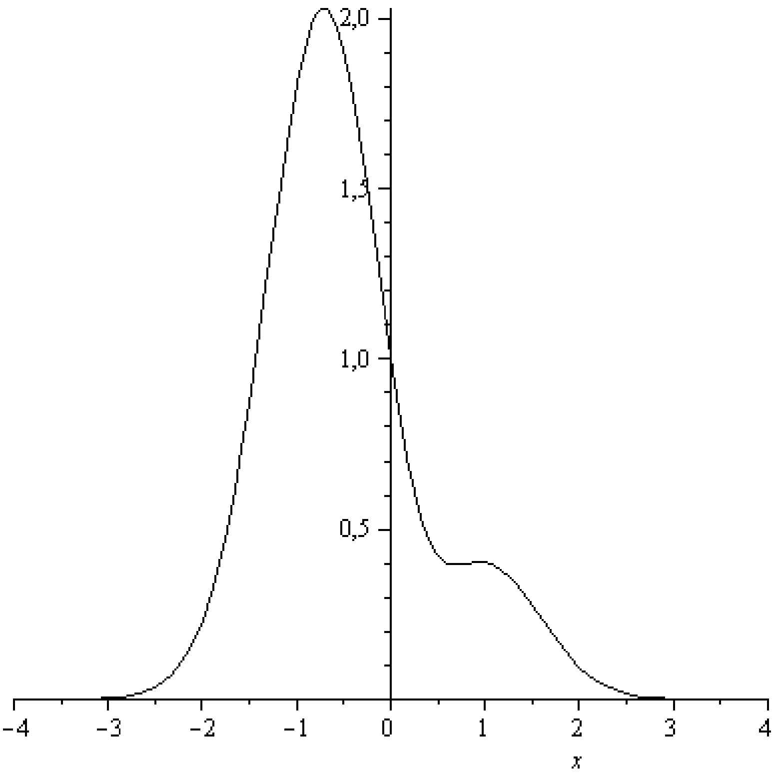
Зависимость $2x_0\sqrt{\pi}\left|\psi\left(x,t_n\right)\right|^2$ $\tau_n = 0.3$



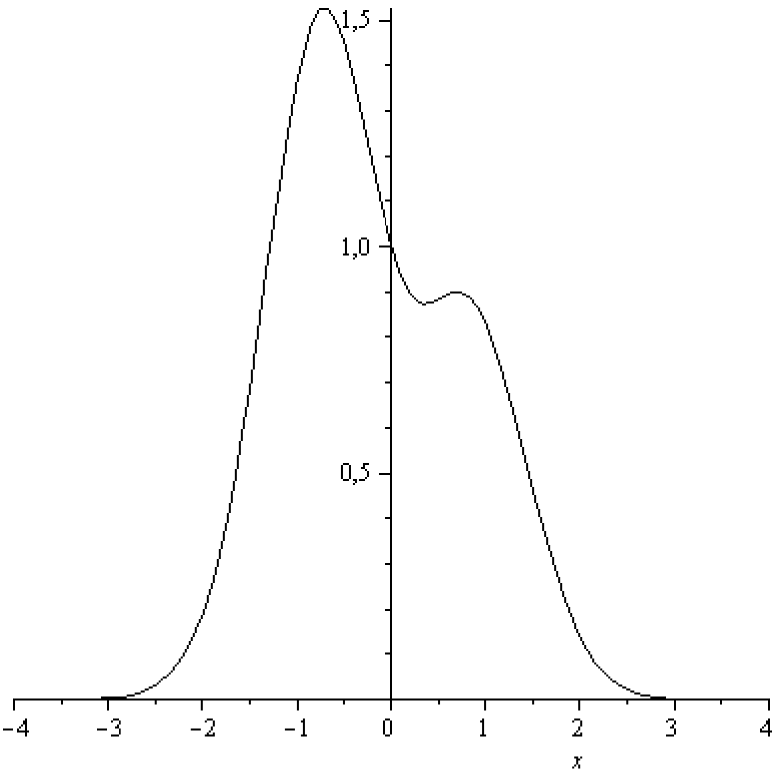
Зависимость $2x_0\sqrt{\pi}\left|\boldsymbol{\psi}\left(x,t_n\right)\right|^2$ $\tau_n = 0.4$



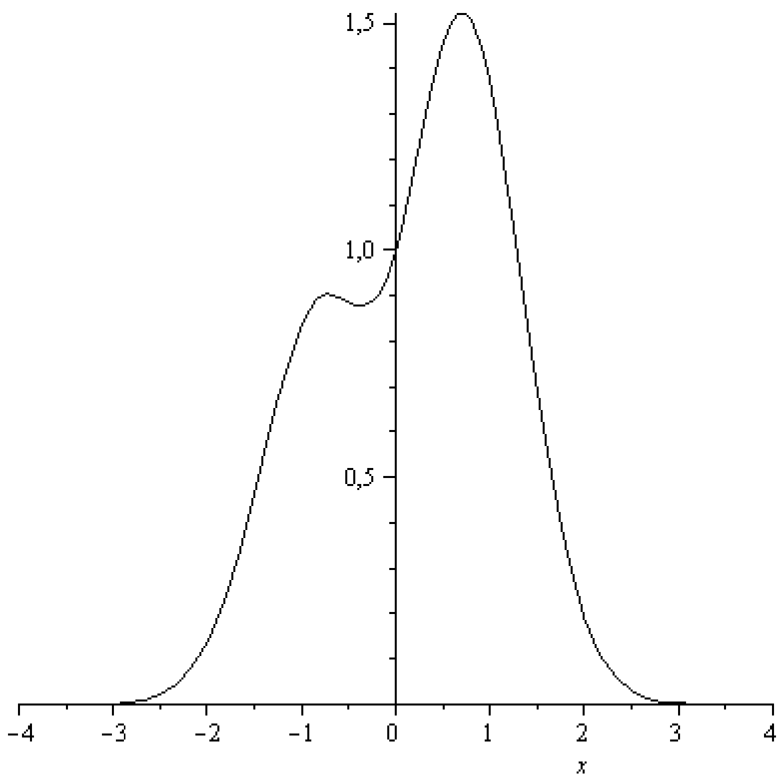
Зависимость $2x_0\sqrt{\pi}\left|\boldsymbol{\psi}\left(x,t_n\right)\right|^2 \quad \tau_n = 0.5$



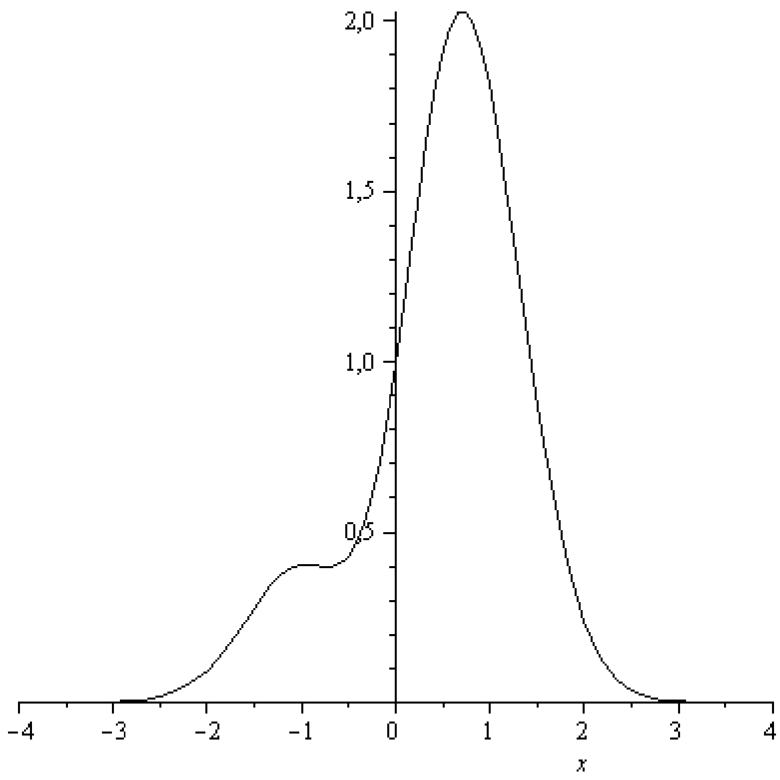
Зависимость $2x_0\sqrt{\pi}\left|\boldsymbol{\psi}\left(x,t_n\right)\right|^2 \quad \tau_n = 0.6$



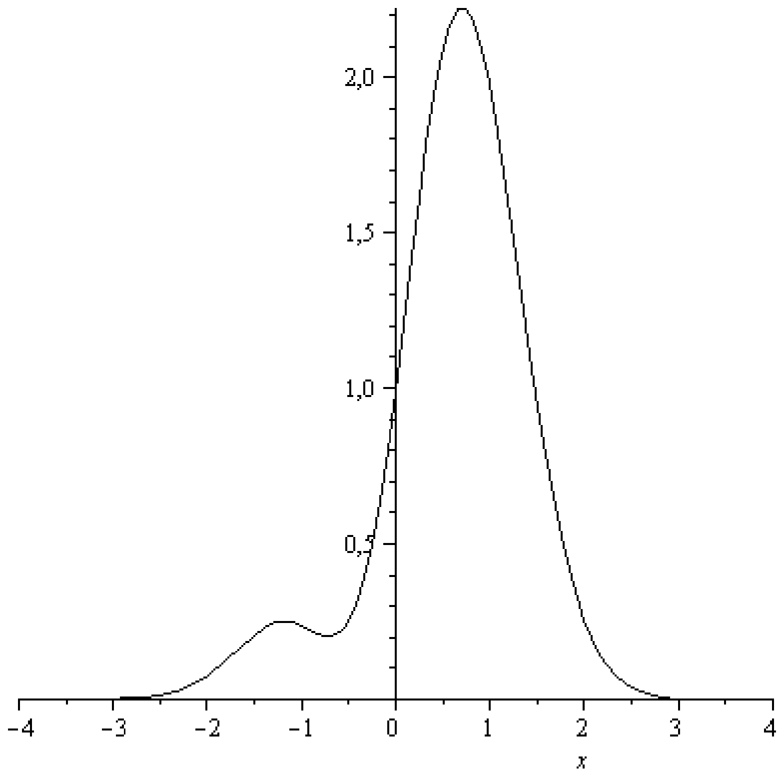
Зависимость $2x_0\sqrt{\pi}\left|\boldsymbol{\psi}\left(x,t_n\right)\right|^2 \quad \tau_n = 0.7$



Зависимость $2x_0\sqrt{\pi}\left|\boldsymbol{\psi}\left(x,t_n\right)\right|^2 \quad \tau_n = 0.8$



Зависимость $2x_0\sqrt{\pi}\left|\psi\left(x,t_n\right)\right|^2 \quad \tau_n = 0.9$



Зависимость $2x_0\sqrt{\pi}\left|\boldsymbol{\psi}\left(x,t_n\right)\right|^2 \quad \tau_n = 1.0$

Расчёт волновой функции 2 возбуждённого состоянии

Вспомним

$$\psi_1(x)=\frac{1}{\sqrt{2x_0\sqrt{\pi}}}\left(2\left(\frac{x}{x_0}\right)\right)\exp\left(-\frac{1}{2}\left(\frac{x}{x_0}\right)^2\right),$$ – волновая функция 1 возбуждённого состояния, $x_0=\sqrt{\frac{\hbar}{m\omega_0}}$

$$\hat{a}^+\left|n\right\rangle=\left|n+1\right\rangle\sqrt{1+n},$$

Если $n=0$, то $\hat{a}^+\psi_1(x)=\sqrt{2}\psi_2(x)$

$$\hat{a}^+=\left(\hat{Q}-i\hat{P}\right)\frac{1}{\sqrt{2}}$$

$$\hat{Q}=\hat{x}\sqrt{\frac{m\omega_0}{\hbar}},\quad \hat{P}=\hat{p}\frac{1}{\sqrt{\hbar m\omega_0}}$$

$$\hat{x}=x,\quad \hat{p}=\frac{\hbar}{i}\frac{\partial}{\partial x}$$

$$\hat{a}^+=\frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega_0}{\hbar}}x-\hbar\frac{1}{\sqrt{\hbar m\omega_0}}\frac{\partial}{\partial x}\right)$$

$$\psi_2(x)=\frac{1}{\sqrt{8\cdot x_0\sqrt{\pi}}}\left(4\left(\frac{x}{x_0}\right)^2-2\right)\exp\left(-\frac{1}{2}\left(\frac{x}{x_0}\right)^2\right)$$

$$\psi_0(x)=\frac{1}{\sqrt{x_0\sqrt{\pi}}}\exp\left(-\frac{1}{2}\left(\frac{x}{x_0}\right)^2\right)$$

$$\psi_2(x)=\frac{1}{\sqrt{8\cdot x_0\sqrt{\pi}}}\left(4\left(\frac{x}{x_0}\right)^2-2\right)\exp\left(-\frac{1}{2}\left(\frac{x}{x_0}\right)^2\right)$$

Волновая функция нестационарного состояния

$$\boldsymbol{\psi}(x,t)=c_1\psi_0(x)e^{-i\frac{E_0t}{\hbar}}+c_2\psi_2(x)e^{-i\frac{E_2t}{\hbar}},\ \ E_0=\frac{\hbar\omega_0}{2},\ \ E_2=\frac{5\hbar\omega_0}{2}$$

$$c_1=c_2=\frac{1}{\sqrt{2}}$$

$$\boldsymbol{\psi}(x,t)=\frac{1}{\sqrt{2}}\left(\psi_0(x)e^{-i\frac{E_0t}{\hbar}}+\psi_2(x)e^{-i\frac{E_2t}{\hbar}}\right)$$

$$\left|\boldsymbol{\psi}(x,t)\right|^2=\frac{1}{2}\left(\psi_0^2(x)+\psi_2^2(x)+\psi_0(x)\psi_2(x)\left(e^{-i\frac{(E_0-E_2)t}{\hbar}}+e^{i\frac{(E_0-E_2)t}{\hbar}}\right)\right)$$

$$E_1-E_0=\hbar\omega_0$$

$$\left|\boldsymbol{\psi}(x,t)\right|^2=\frac{1}{2}\left(\psi_0^2(x)+\psi_2^2(x)+2\psi_0(x)\psi_2(x)\cos(2\omega_0t)\right)$$

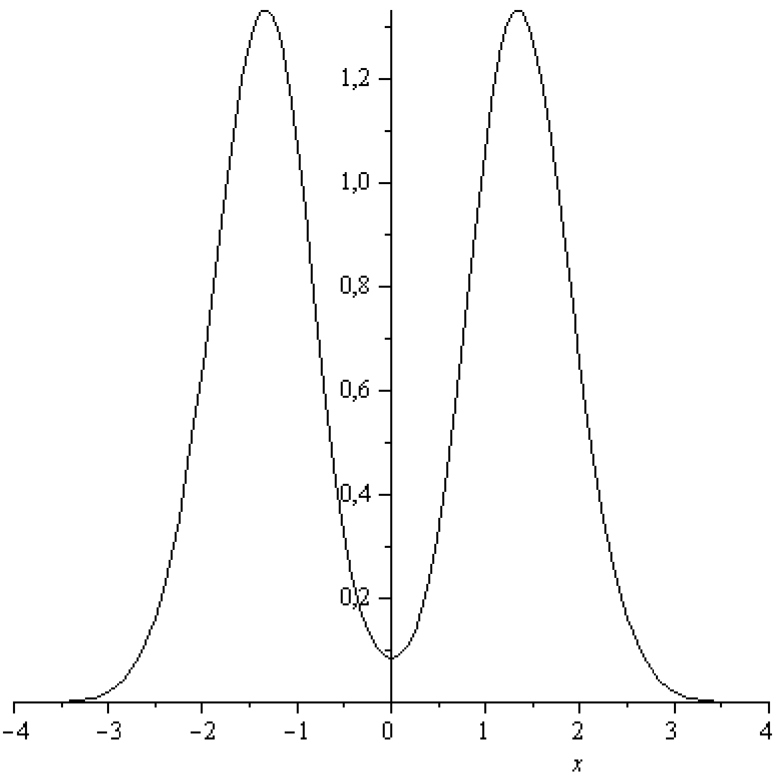
$$2\omega_0t_n=\tau_n$$

$$\tau_n=2\pi\cdot 0;2\pi\cdot\frac{1}{10};2\pi\cdot\frac{2}{10};2\pi\cdot\frac{3}{10};...2\pi\cdot\frac{10}{10};$$

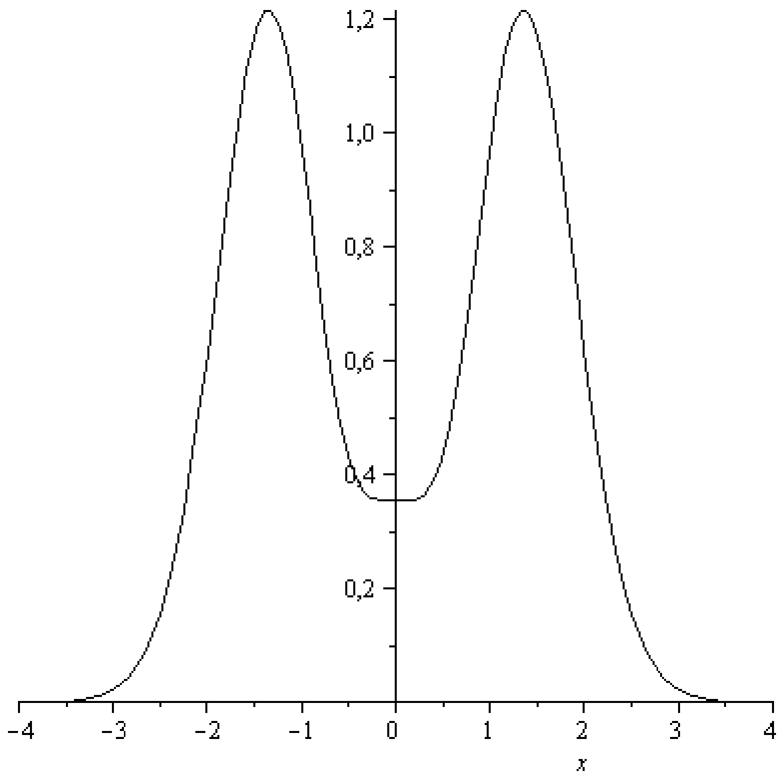
$$\frac{x}{x_0}=x'$$

$$2x_0\sqrt{\pi}\left|\boldsymbol{\psi}(x,t)\right|^2=$$

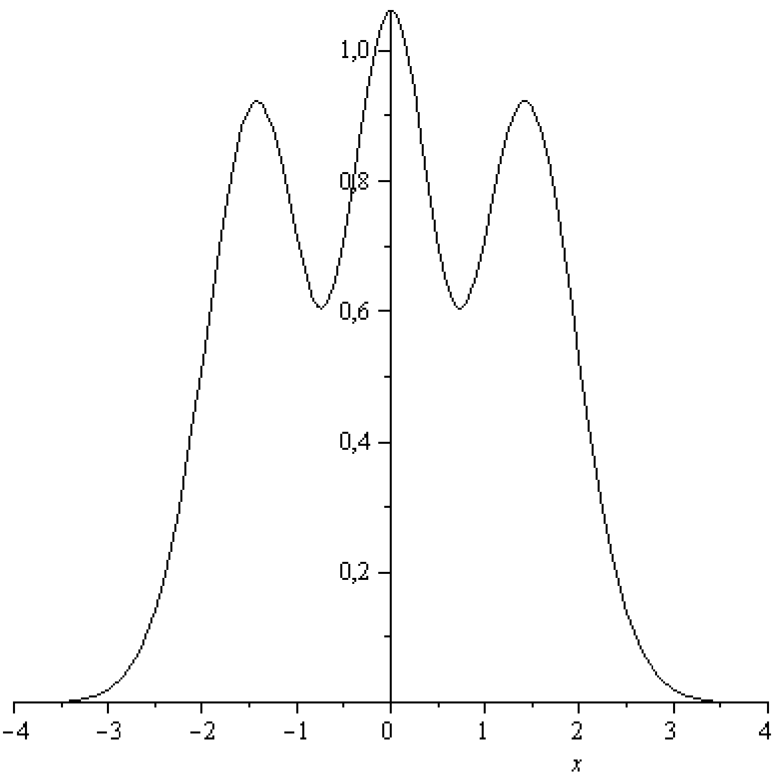
$$=\left(1+\left(\frac{1}{\sqrt{8}}\left(4x'^2-2\right)\right)^2+2\left(\frac{1}{\sqrt{8}}\left(4x'^2-2\right)\right)\cos(2\omega_0t)\right)\exp\left(-x'^2\right)$$



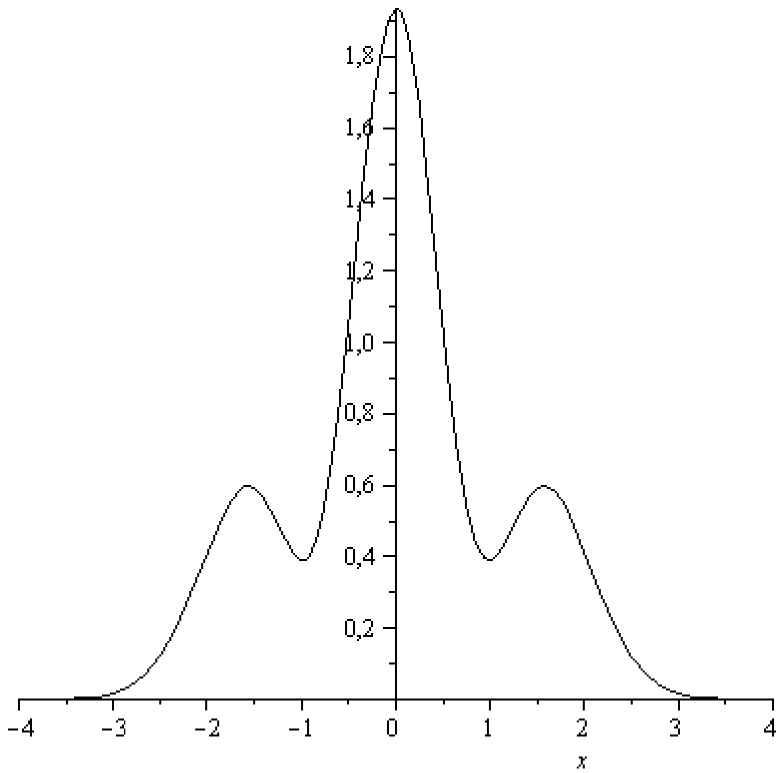
Зависимость $2x_0\sqrt{\pi}\left|\boldsymbol{\psi}\left(x,t_n\right)\right|^2 \quad \tau_n = 0$



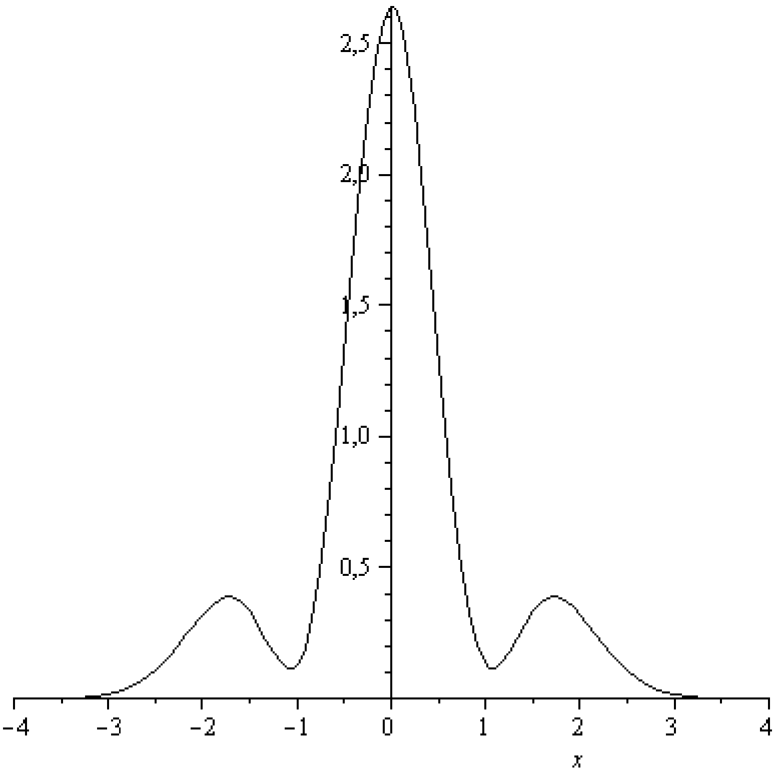
Зависимость $2x_0\sqrt{\pi}\left|\boldsymbol{\psi}\left(x,t_n\right)\right|^2 \quad \tau_n = 0.1$



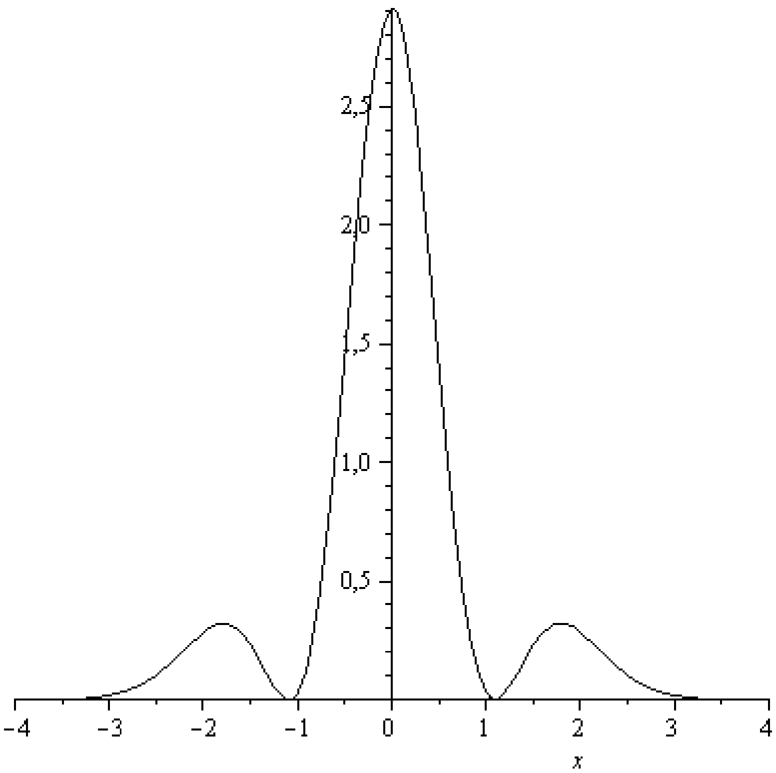
Зависимость $2x_0\sqrt{\pi}\left|\boldsymbol{\psi}\left(x,t_n\right)\right|^2 \quad \tau_n = 0.2$



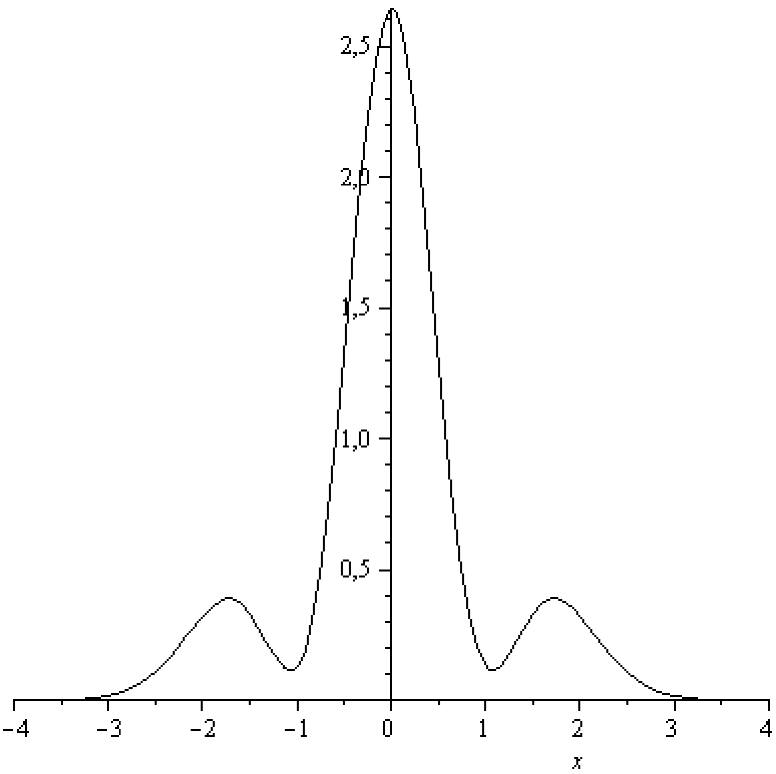
Зависимость $2x_0\sqrt{\pi}\left|\boldsymbol{\psi}\left(x,t_n\right)\right|^2 \quad \tau_n = 0.3$



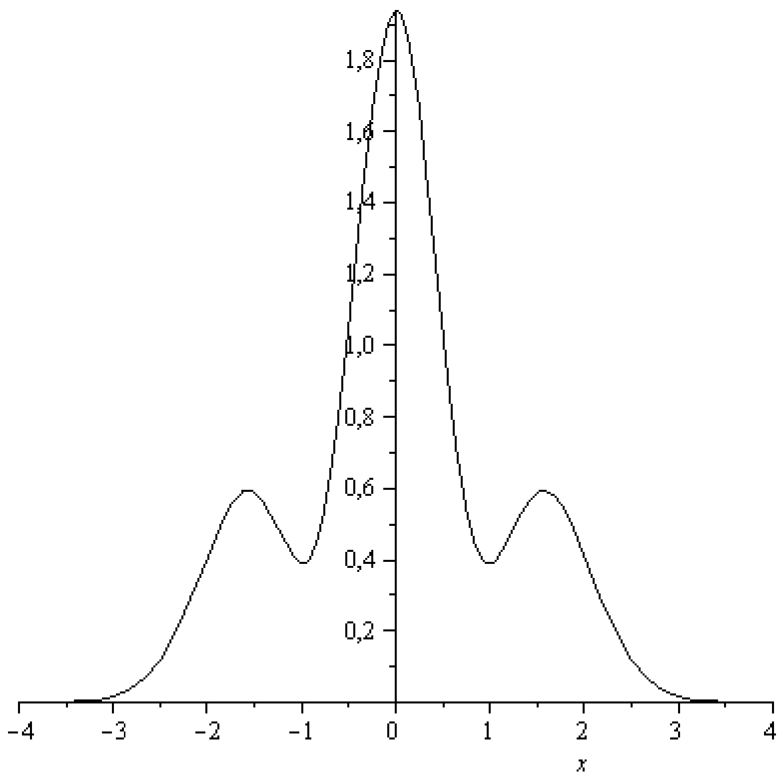
Зависимость $2x_0\sqrt{\pi}\left|\boldsymbol{\psi}\left(x,t_n\right)\right|^2$ $\tau_n = 0.4$



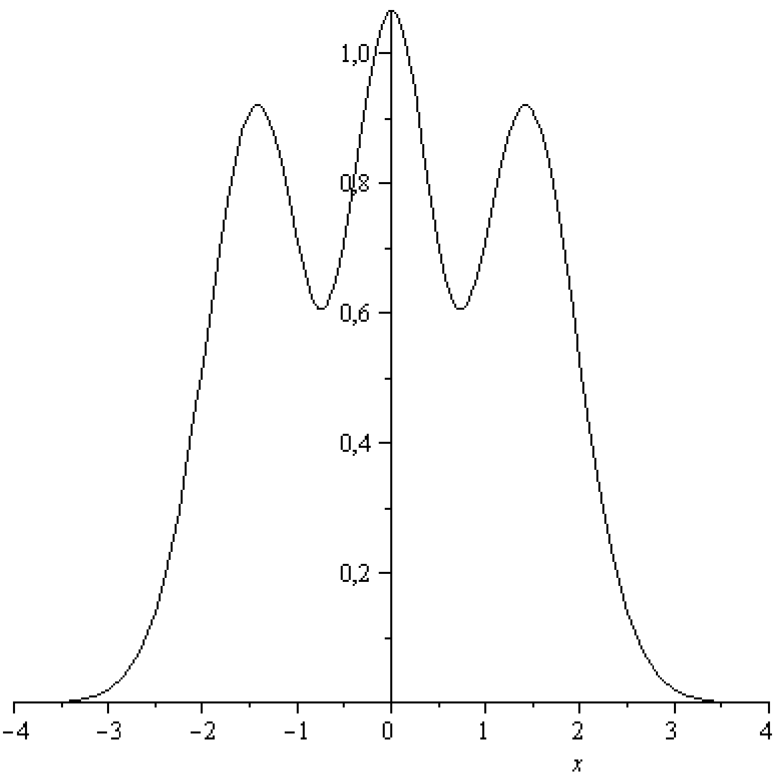
Зависимость $2x_0\sqrt{\pi}\left|\boldsymbol{\psi}\left(x,t_n\right)\right|^2 \quad \tau_n = 0.5$



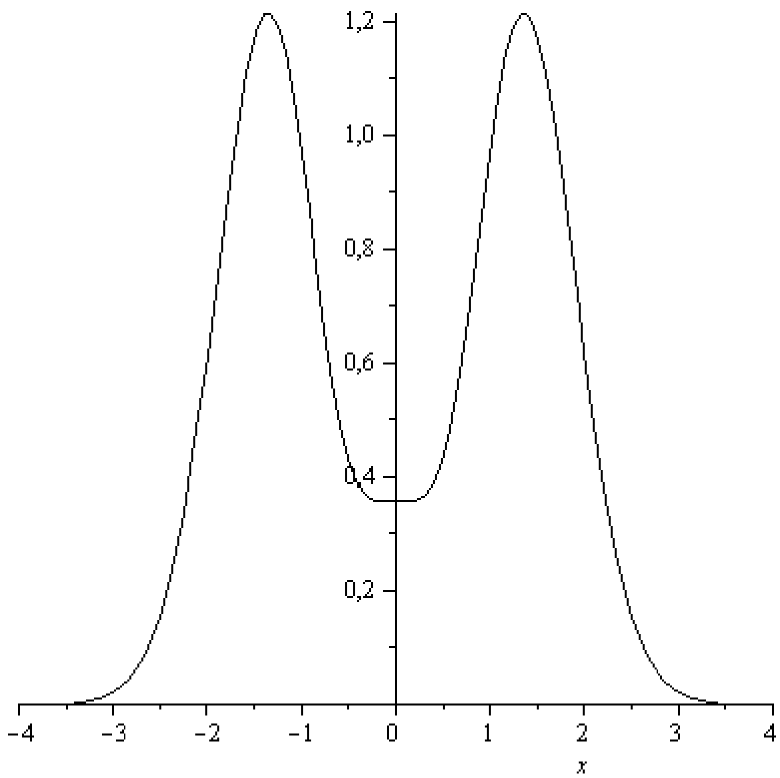
Зависимость $2x_0\sqrt{\pi}\left|\boldsymbol{\psi}\left(x,t_n\right)\right|^2\quad \tau_n = 0.6$



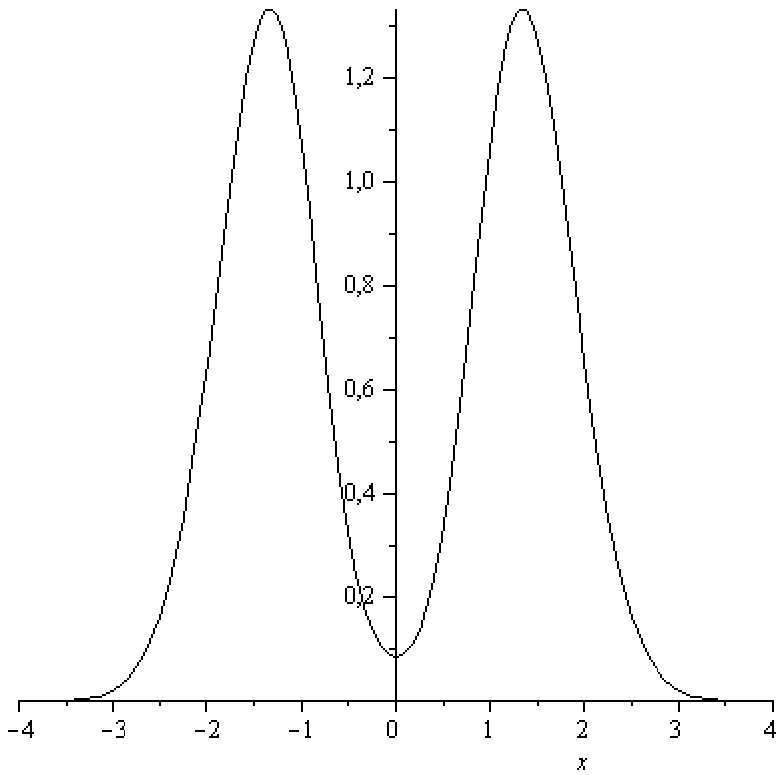
Зависимость $2x_0\sqrt{\pi}\left|\boldsymbol{\psi}\left(x,t_n\right)\right|^2$ $\tau_n = 0.7$



Зависимость $2x_0\sqrt{\pi}\left|\boldsymbol{\psi}\left(x,t_n\right)\right|^2 \quad \tau_n = 0.8$



Зависимость $2x_0\sqrt{\pi}\left|\boldsymbol{\psi}\left(x,t_n\right)\right|^2 \quad \tau_n = 0.9$



Зависимость $2x_0\sqrt{\pi}\left|\boldsymbol{\psi}\left(x,t_n\right)\right|^2 \quad \tau_n = 1$