1.8 PARAMETRIC EQUATIONS

Thus far, our study of graphs has focused on graphs of functions. However, because such graphs must pass the vertical line test, this limitation precludes curves with self-intersections or even such basic curves as circles. In this section we will study an alternative method for describing curves algebraically that is not subject to the severe restriction of the vertical line test.

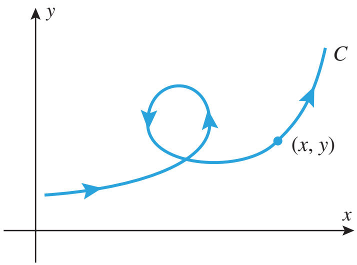
This material is placed here to provide an early parametric option. However, it can be deferred until Chapter 11, if preferred.

PARAMETRIC EQUATIONS

Suppose that a particle moves along a curve C in the xy-plane in such a way that its x- and y-coordinates, as functions of time, are

$$x = f(t), \quad y = g(t)$$

We call these the ***parametric equations*** of motion for the particle and refer to C as the ***trajectory*** of the particle or the ***graph*** of the equations (Figure 1.8.1). The variable t is called the ***parameter*** for the equations.



A moving particle with trajectory C

Figure 1.8.1

▶ **Example 1** Sketch the trajectory over the time interval $0 \le t \le 10$ of the particle whose parametric equations of motion are

$$x = t - 3\sin t, \quad y = 4 - 3\cos t \tag{1}$$

Solution. One way to sketch the trajectory is to choose a representative succession of times, plot the (x, y) coordinates of points on the trajectory at those times, and connect the points with a smooth curve. The trajectory in Figure 1.8.2 was obtained in this way from the data in Table 1.8.1 in which the approximate coordinates of the particle are given at time increments of 1 unit. Observe that there is no t-axis in the picture; the values of t appear only as labels on the plotted points, and even these are usually omitted unless it is important to emphasize the locations of the particle at specific times. ◀

TECHNOLOGY MASTERY

Read the documentation for your graphing utility to learn how to graph parametric equations, and then generate the trajectory in Example 1. Explore the behavior of the particle beyond time $t = 10$.

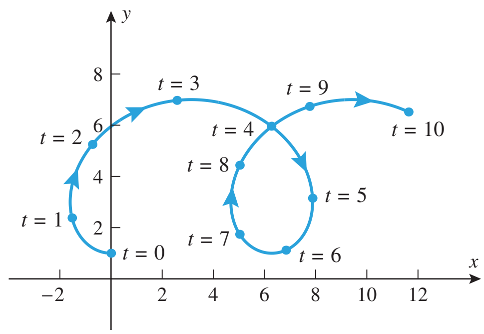


Figure 1.8.2

Table 1.8.1

t	x	y
0	0.0	1.0
1	−1.5	2.4
2	−0.7	5.2
3	2.6	7.0
4	6.3	6.0
5	7.9	3.1
6	6.8	1.1
7	5.0	1.7
8	5.0	4.4
9	7.8	6.7
10	11.6	6.5

Although parametric equations commonly arise in problems of motion with time as the parameter, they arise in other contexts as well. Thus, unless the problem dictates that the parameter t in the equations

$$x = f(t), \quad y = g(t)$$

represents time, it should be viewed simply as an independent variable that varies over some interval of real numbers. (In fact, there is no need to use the letter t for the parameter; any letter not reserved for another purpose can be used.) If no restrictions on the parameter are stated explicitly or implied by the equations, then it is understood that it varies from $-\infty$ to $+\infty$. To indicate that a parameter t is restricted to an interval $[a, b]$, we will write

$$x = f(t), \quad y = g(t) \quad (a \le t \le b)$$

▶ **Example 2** Find the graph of the parametric equations

$$x = \cos t, \quad y = \sin t \quad (0 \le t \le 2\pi) \tag{2}$$

Solution. One way to find the graph is to eliminate the parameter t by noting that

$$x^2 + y^2 = \sin^2 t + \cos^2 t = 1$$

Thus, the graph is contained in the unit circle $x^2 + y^2 = 1$. Geometrically, the parameter t can be interpreted as the angle swept out by the radial line from the origin to the point $(x, y) = (\cos t, \sin t)$ on the unit circle (Figure 1.8.3). As t increases from 0 to 2π, the point traces the circle counterclockwise, starting at $(1, 0)$ when $t = 0$ and completing one full revolution when $t = 2\pi$. One can obtain different portions of the circle by varying the interval over which the parameter varies. For example,

$$x = \cos t, \quad y = \sin t \quad (0 \le t \le \pi) \tag{3}$$

represents just the upper semicircle in Figure 1.8.3. ◀

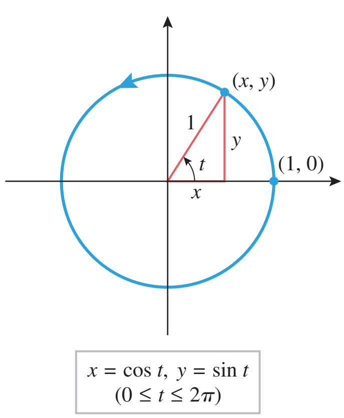


Figure 1.8.3

ORIENTATION

The direction in which the graph of a pair of parametric equations is traced as the parameter increases is called the ***direction of increasing parameter*** or sometimes the ***orientation*** imposed on the curve by the equations. Thus, we make a distinction between a ***curve***, which is a set of points, and a ***parametric curve***, which is a curve with an orientation imposed on it by a set of parametric equations. For example, we saw in Example 2 that the circle represented parametrically by (2) is traced counterclockwise as t increases and hence has *counterclockwise orientation*. As shown in Figures 1.8.2 and 1.8.3, the orientation of a parametric curve can be indicated by arrowheads.

To obtain parametric equations for the unit circle with *clockwise orientation*, we can replace t by $-t$ in (2) and use the identities $\cos(-t) = \cos t$ and $\sin(-t) = -\sin t$. This yields

$$x = \cos t, \quad y = -\sin t \qquad (0 \le t \le 2\pi)$$

Here, the circle is traced clockwise by a point that starts at $(1, 0)$ when $t = 0$ and completes one full revolution when $t = 2\pi$ (Figure 1.8.4).

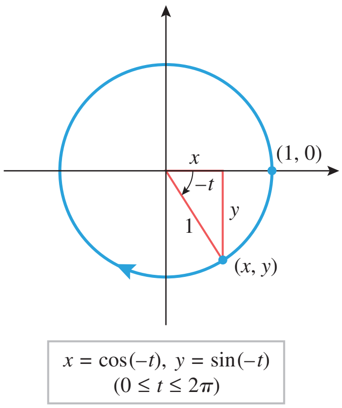


Figure 1.8.4

TECHNOLOGY MASTERY

When parametric equations are graphed using a calculator, the orientation can often be determined by watching the direction in which the graph is traced on the screen. However, many computers graph so fast that it is often hard to discern the orientation. See if you can use your graphing utility to confirm that (3) has a counterclockwise orientation.

▶ Example 3 Graph the parametric curve

$$x = 2t - 3, \quad y = 6t - 7$$

by eliminating the parameter, and indicate the orientation on the graph.

Solution. To eliminate the parameter we will solve the first equation for t as a function of x, and then substitute this expression for t into the second equation:

$$t = \left(\tfrac{1}{2}\right)(x + 3)$$

$$y = 6\left(\tfrac{1}{2}\right)(x + 3) - 7$$

$$y = 3x + 2$$

Thus, the graph is a line of slope 3 and y-intercept 2. To find the orientation we must look to the original equations; the direction of increasing t can be deduced by observing that x increases as t increases *or* by observing that y increases as t increases. Either piece of information tells us that the line is traced left to right as shown in Figure 1.8.5. ◀

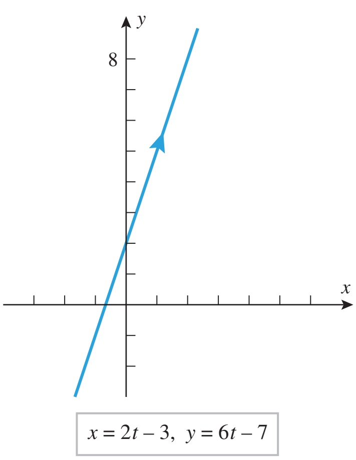


Figure 1.8.5

Not all parametric equations produce curves with definite orientations; if the equations are badly behaved, then the point tracing the curve may leap around sporadically or move back and forth, failing to determine a definite direction. For example, if

$$x = \sin t, \quad y = \sin^2 t$$

then the point (x, y) moves along the parabola $y = x^2$. However, the value of x varies periodically between -1 and 1, so the point (x, y) moves periodically back and forth along the parabola between the points $(-1, 1)$ and $(1, 1)$ (as shown in Figure 1.8.6). Later in the text we will discuss restrictions that eliminate such erratic behavior, but for now we will just avoid such complications.

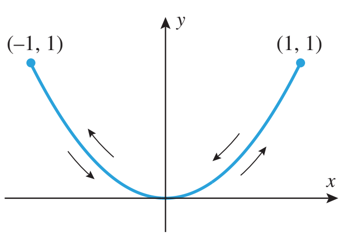


Figure 1.8.6

EXPRESSING ORDINARY FUNCTIONS PARAMETRICALLY

An equation $y = f(x)$ can be expressed in parametric form by introducing the parameter $t = x$; this yields the parametric equations $x = t$, $y = f(t)$. For example, the portion of the curve $y = \cos x$ over the interval $[-2\pi, 2\pi]$ can be expressed parametrically as

$$x = t, \quad y = \cos t \qquad (-2\pi \le t \le 2\pi)$$

(Figure 1.8.7).

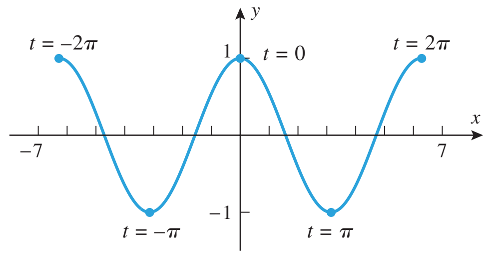


Figure 1.8.7

GENERATING PARAMETRIC CURVES WITH GRAPHING UTILITIES

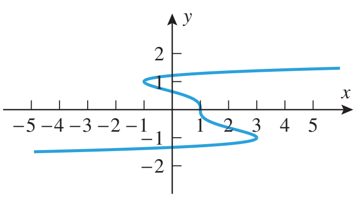
Many graphing utilities allow you to graph equations of the form $y = f(x)$ but not equations of the form $x = g(y)$. Sometimes you will be able to rewrite $x = g(y)$ in the form $y = f(x)$; however, if this is inconvenient or impossible, then you can graph $x = g(y)$ by introducing a parameter $t = y$ and expressing the equation in the parametric form $x = g(t)$, $y = t$. (You may have to experiment with various intervals for t to produce a complete graph.)

Example 4 Use a graphing utility to graph the equation $x = 3y^5 - 5y^3 + 1$.

Solution. If we let $t = y$ be the parameter, then the equation can be written in parametric form as

$$x = 3t^5 - 5t^3 + 1, \quad y = t$$

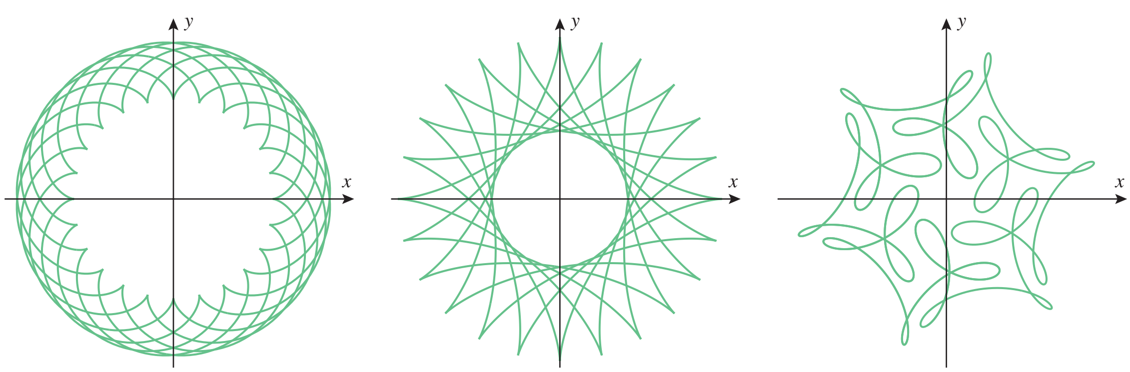
Figure 1.8.8 shows the graph of these equations for $-1.5 \le t \le 1.5$. ◀



$x = 3t^5 - 5t^3 + 1,\ y = t$
$-1.5 \le t \le 1.5$

Figure 1.8.8

Some parametric curves are so complex that it is virtually impossible to visualize them without using some kind of graphing utility. Figure 1.8.9 shows three such curves.



$x = 31\cos t - 7\cos(31/7)t$
$y = 31\sin t - 7\sin(31/7)t$
$(0 \le t \le 14\pi)$

$x = 17\cos t + 7\cos(17/7)t$
$y = 17\sin t - 7\sin(17/7)t$
$(0 \le t \le 14\pi)$

$x = \cos t + (1/2)\cos 7t + (1/3)\sin 17t$
$y = \sin t + (1/2)\sin 7t + (1/3)\cos 17t$
$(0 \le t \le 2\pi)$

Figure 1.8.9

GRAPHING INVERSE FUNCTIONS WITH GRAPHING UTILITIES

TECHNOLOGY MASTERY

Try your hand at using a graphing utility to generate some parametric curves that you think are interesting or beautiful.

Most graphing utilities cannot graph inverse functions directly. However, there is a way of graphing inverse functions by expressing the graphs parametrically. To see how this can be done, suppose that we are interested in graphing the inverse of a one-to-one function f. We know that the equation $y = f(x)$ can be expressed parametrically as

$$x = t, \quad y = f(t) \tag{4}$$

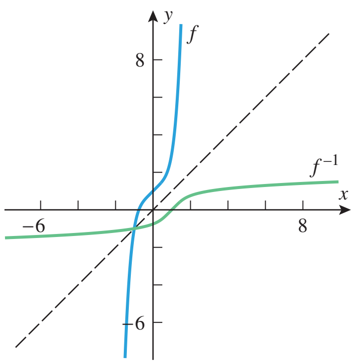


Figure 1.8.10

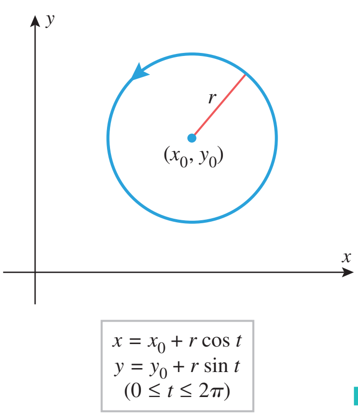


Figure 1.8.11

TECHNOLOGY MASTERY

Use the parametric capability of your graphing utility to generate a circle of radius 5 that is centered at $(3, -2)$.

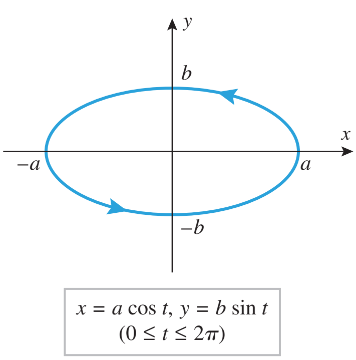


Figure 1.8.12

and we know that the graph of f^{-1} can be obtained by interchanging x and y, since this reflects the graph of f about the line $y = x$. Thus, from (4) the graph of f^{-1} can be represented parametrically as

$$x = f(t), \quad y = t \tag{5}$$

For example, Figure 1.8.10 shows the graph of $f(x) = x^5 + x + 1$ and its inverse generated with a graphing utility. The graph of f was generated from the parametric equations

$$x = t, \quad y = t^5 + t + 1$$

and the graph of f^{-1} was generated from the parametric equations

$$x = t^5 + t + 1, \quad y = t$$

TRANSLATION

If a parametric curve C is given by the equations $x = f(t)$, $y = g(t)$, then adding a constant to $f(t)$ translates the curve C in the x-direction, and adding a constant to $g(t)$ translates it in the y-direction. Thus, a circle of radius r, centered at (x_0, y_0) can be represented parametrically as

$$x = x_0 + r\cos t, \quad y = y_0 + r\sin t \qquad (0 \le t \le 2\pi) \tag{6}$$

(Figure 1.8.11). If desired, we can eliminate the parameter from these equations by noting that

$$(x - x_0)^2 + (y - y_0)^2 = (r\cos t)^2 + (r\sin t)^2 = r^2$$

Thus, we have obtained the familiar equation in rectangular coordinates for a circle of radius r, centered at (x_0, y_0):

$$(x - x_0)^2 + (y - y_0)^2 = r^2 \tag{7}$$

SCALING

If a parametric curve C is given by the equations $x = f(t)$, $y = g(t)$, then multiplying $f(t)$ by a constant stretches or compresses C in the x-direction, and multiplying $g(t)$ by a constant stretches or compresses C in the y-direction. For example, we would expect the parametric equations

$$x = 3\cos t, \quad y = 2\sin t \qquad (0 \le t \le 2\pi)$$

to represent an ellipse, centered at the origin, since the graph of these equations results from stretching the unit circle

$$x = \cos t, \quad y = \sin t \qquad (0 \le t \le 2\pi)$$

by a factor of 3 in the x-direction and a factor of 2 in the y-direction. In general, if a and b are positive constants, then the parametric equations

$$x = a\cos t, \quad y = b\sin t \qquad (0 \le t \le 2\pi) \tag{8}$$

represent an ellipse, centered at the origin, and extending between $-a$ and a on the x-axis and between $-b$ and b on the y-axis (Figure 1.8.12). The numbers a and b are called the ***semiaxes*** of the ellipse. If desired, we can eliminate the parameter t in (8) and rewrite the equations in rectangular coordinates as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{9}$$

TECHNOLOGY MASTERY

Use the parametric capability of your graphing utility to generate an ellipse that is centered at the origin and that extends between -4 and 4 in the x-direction and between -3 and 3 in the y-direction. Generate an ellipse with the same dimensions, but translated so that its center is at the point $(2, 3)$.

LISSAJOUS CURVES

In the mid-1850s the French physicist Jules Antoine Lissajous (1822–1880) became interested in parametric equations of the form

$$x = \sin at, \quad y = \sin bt \tag{10}$$

in the course of studying vibrations that combine two perpendicular sinusoidal motions. The first equation in (10) describes a sinusoidal oscillation in the x-direction with frequency $a/2\pi$, and the second describes a sinusoidal oscillation in the y-direction with frequency $b/2\pi$. If a/b is a rational number, then the combined effect of the oscillations is a periodic motion along a path called a ***Lissajous curve***. Figure 1.8.13 shows some typical Lissajous curves.

TECHNOLOGY MASTERY

Generate some Lissajous curves on your graphing utility, and also see if you can figure out when each of the curves in Figure 1.8.13 begins to repeat.

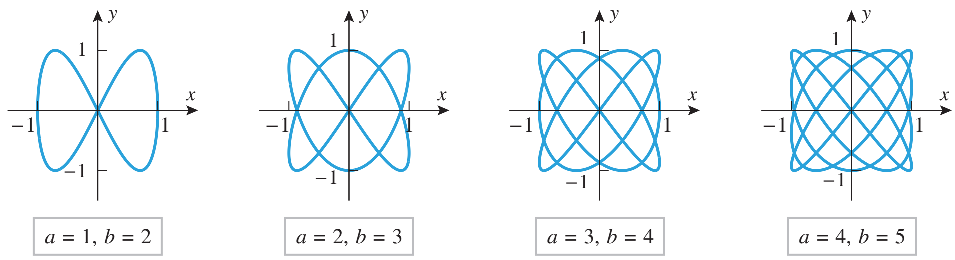


Figure 1.8.13

CYCLOIDS

If a wheel rolls in a straight line along a flat road, then a point on the rim of the wheel will trace a curve called a ***cycloid*** (Figure 1.8.14). This curve has a fascinating history, which we will discuss shortly; but first we will show how to obtain parametric equations for it. For this purpose, let us assume that the wheel has radius a and rolls along the positive x-axis of a rectangular coordinate system. Let $P(x, y)$ be the point on the rim that traces the cycloid, and assume that P is initially at the origin. We will take as our parameter the angle θ that is swept out by the radial line to P as the wheel rolls (Figure 1.8.14). It is standard here to regard θ to be positive, even though it is generated by a clockwise rotation.

The motion of P is a combination of the movement of the wheel's center parallel to the x-axis and the rotation of P around the center. As the radial line sweeps out an angle θ, the point P traverses an arc of length $a\theta$, and the wheel moves a distance $a\theta$ along the x-axis (why?). Thus, as suggested by Figure 1.8.15, the center moves to the point $(a\theta, a)$, and the coordinates of $P(x, y)$ are

TECHNOLOGY MASTERY

Use your graphing utility to generate two "arches" of the cycloid produced by a point on the rim of a wheel of radius 1.

$$x = a\theta - a\sin\theta, \quad y = a - a\cos\theta \tag{11}$$

These are the equations of the cycloid in terms of the parameter θ.

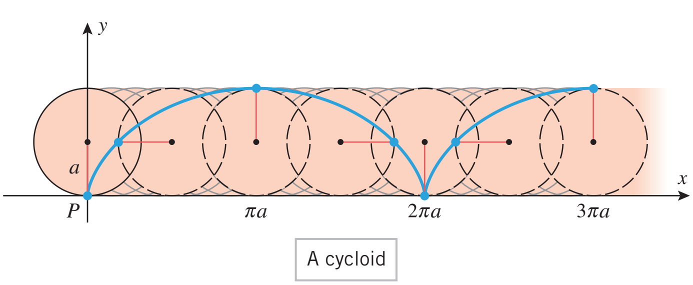


Figure 1.8.14

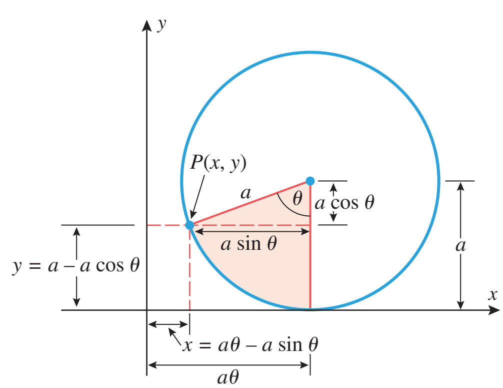


Figure 1.8.15

THE ROLE OF THE CYCLOID IN MATHEMATICS HISTORY

The cycloid is of interest because it provides the solution to two famous mathematical problems—the ***brachistochrone problem*** (from Greek words meaning "shortest time") and the ***tautochrone problem*** (from Greek words meaning "equal time"). The brachistochrone problem is to determine the shape of a wire along which a bead might slide from a point P to another point Q, not directly below, in the *shortest time*. The tautochrone problem is to find the shape of a wire from P to Q such that two beads started at any points on the wire between P and Q reach Q in the same amount of time. The solution to both problems turns out to be an inverted cycloid (Figure 1.8.16).

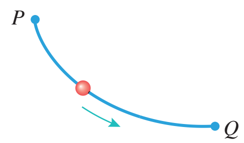


Figure 1.8.16

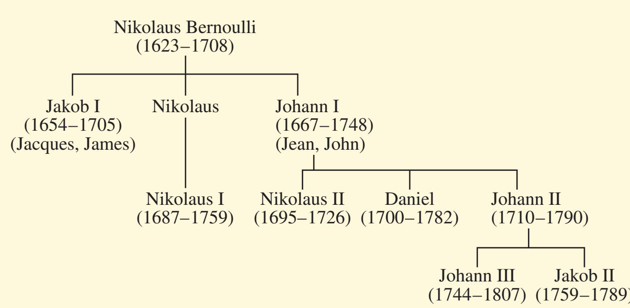
In June of 1696, Johann Bernoulli posed the brachistochrone problem in the form of a challenge to other mathematicians. At first, one might conjecture that the wire should form a straight line, since that shape results in the shortest distance from P to Q. However, the inverted cycloid allows the bead to fall more rapidly at first, building up sufficient initial speed to reach Q in the shortest time, even though it travels a longer distance. The problem was solved by Newton and Leibniz as well as by Johann Bernoulli and his older brother Jakob; it was formulated and solved *incorrectly* years earlier by Galileo, who thought the answer was a circular arc.

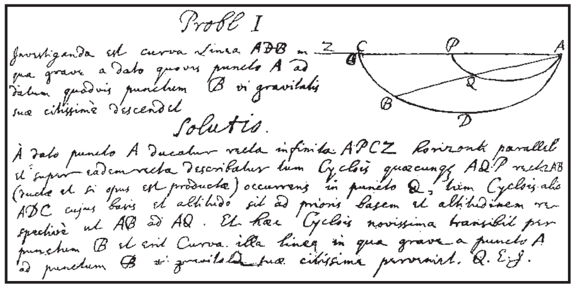
Bernoulli An amazing Swiss family that included several generations of outstanding mathematicians and scientists. Nikolaus Bernoulli (1623–1708), a druggist, fled from Antwerp to escape religious persecution and ultimately set tled in Basel, Switzerland. There he had three sons, Jakob I (also called Jacques or James), Nikolaus, and Johann I (also called Jean or John). The Roman numerals are used to distinguish family members with identical names (see the family tree below). Following Newton and Leibniz, the Bernoulli brothers, Jakob I and Johann I, are considered by some to be the two most important founders of calculus. Jakob I was self-taught in mathematics. His father wanted him to study for the ministry, but he turned to mathematics and in 1686 became a professor at the University of Basel. When he started working in mathematics, he knew nothing of Newton's and Leibniz' work. He eventually became familiar with Newton's results, but because so little of Leibniz' work was published, Jakob duplicated many of Leibniz' results.

Jakob's younger brother Johann I was urged to enter into business by his father. Instead, he turned to medicine and studied mathematics under the guidance of his older brother. He eventually became a mathematics professor at Gröningen in Holland, and then, when Jakob died in 1705, Johann succeeded him as mathematics professor at Basel. Throughout their lives, Jakob I and Johann I had a mutual passion for criticizing each other's work, which frequently erupted into ugly confrontations. Leibniz tried to mediate the disputes, but Jakob, who resented Leibniz' superior intellect, accused him of siding with Johann, and thus Leibniz became entangled in the arguments. The brothers often worked on common problems that they posed as challenges to one another. Johann, interested in gaining fame, often used unscrupulous means to make himself appear the originator of his brother's results; Jakob occasionally retaliated. Thus, it is often difficult to determine who deserves credit for many results. However, both men made major contributions to the development of calculus. In addition to his work on calculus, Jakob helped establish fundamental principles in probability, including the Law of Large Numbers, which is a cornerstone of modern probability theory.

Among the other members of the Bernoulli family, Daniel, son of Johann I, is the most famous. He was a professor of mathematics at St. Petersburg Academy in Russia and subsequently a professor of anatomy and then physics at Basel. He did work in calculus and probability, but is best known for his work in physics. A basic law of fluid flow, called Bernoulli's principle, is named in his honor. He won the annual prize of the French Academy 10 times for work on vibrating strings, tides of the sea, and kinetic theory of gases.

Johann II succeeded his father as professor of mathematics at Basel. His research was on the theory of heat and sound. Nikolaus I was a mathematician and law scholar who worked on probability and series. On the recommendation of Leibniz, he was appointed professor of mathematics at Padua and then went to Basel as a professor of logic and then law. Nikolaus II was professor of jurisprudence in Switzerland and then professor of mathematics at St. Petersburg Academy. Johann III was a professor of mathematics and astronomy in Berlin and Jakob II succeeded his uncle Daniel as professor of mathematics at St. Petersburg Academy in Russia. Truly an incredible family!




Probl I

Investiganda est curva Linea ADB in qua grave a dato quovis puncto A ad datum quodvis punctum B vi gravitatis suae citissime descendet.

Solutio.

A dato puncto A ducatur recta infinita APCZ horizonti parallela et super eadem recta describatur tum Cyclois quaecunque AQP rectae AB (ductae et si opus est productae) occurrens in puncto Q, tum Cyclois alia ADC cujus basis et altitudo sit ad prioris basem et altitudinem respective ut AB ad AQ. Et haec Cyclois novissima transibit per punctum B et erit Curva illa linea in qua grave a puncto A ad punctum B vi gravitatis suae citissime perveniet. Q.E.I.

Newton's solution of the brachistochrone problem in his own handwriting

✔ QUICK CHECK EXERCISES 1.8 (See page 96 for answers.)

1. Find parametric equations for a circle of radius 2, centered at (3, 5).
2. Find parametric equations for the ellipse
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
3. The graph of the curve described by the parametric equations $x = 4t - 1$, $y = 3t + 2$ is a straight line with slope ________ and y-intercept ________.
4. Suppose that a parametric curve C is given by the equations $x = f(t)$, $y = g(t)$ for $0 \le t \le 1$. Find parametric equations for C that reverse the direction the curve is traced as the parameter increases from 0 to 1.
5. If $f(x)$ is a one-to-one function, then parametric equations for the curve $y = f^{-1}(x)$ are given by $x =$ ________ and $y = t$.

EXERCISE SET 1.8 Graphing Utility

1. (a) By eliminating the parameter, sketch the trajectory over the time interval $0 \le t \le 5$ of the particle whose parametric equations of motion are
$$x = t - 1, \quad y = t + 1$$
(b) Indicate the direction of motion on your sketch.
(c) Make a table of x- and y-coordinates of the particle at times $t = 0, 1, 2, 3, 4, 5$.
(d) Mark the position of the particle on the curve at the times in part (c), and label those positions with the values of t.
2. (a) By eliminating the parameter, sketch the trajectory over the time interval $0 \le t \le 1$ of the particle whose parametric equations of motion are
$$x = \cos(\pi t), \quad y = \sin(\pi t)$$
(b) Indicate the direction of motion on your sketch.
(c) Make a table of x- and y-coordinates of the particle at times $t = 0, 0.25, 0.5, 0.75, 1$.
(d) Mark the position of the particle on the curve at the times in part (c), and label those positions with the values of t.

3–12 Sketch the curve by eliminating the parameter, and indicate the direction of increasing t.

3. $x = 3t - 4$, $y = 6t + 2$
4. $x = t - 3$, $y = 3t - 7 \quad (0 \le t \le 3)$
5. $x = 2\cos t$, $y = 5\sin t \quad (0 \le t \le 2\pi)$
6. $x = \sqrt{t}$, $y = 2t + 4$
7. $x = 3 + 2\cos t$, $y = 2 + 4\sin t \quad (0 \le t \le 2\pi)$
8. $x = \sec t$, $y = \tan t \quad (\pi \le t < 3\pi/2)$
9. $x = \cos 2t$, $y = \sin t \quad (-\pi/2 \le t \le \pi/2)$
10. $x = 4t + 3$, $y = 16t^2 - 9$
11. $x = 2\sin^2 t$, $y = 3\cos^2 t$
12. $x = \sec^2 t$, $y = \tan^2 t \quad (-\pi/2 < t < \pi/2)$

13–18 Find parametric equations for the curve, and check your work by generating the curve with a graphing utility.

13. A circle of radius 5, centered at the origin, oriented clockwise.
14. The portion of the circle $x^2 + y^2 = 1$ that lies in the third quadrant, oriented counterclockwise.
15. A vertical line intersecting the x-axis at $x = 2$, oriented upward.
16. The ellipse $x^2/4 + y^2/9 = 1$, oriented counterclockwise.
17. The portion of the parabola $x = y^2$ joining $(1, -1)$ and $(1, 1)$, oriented down to up.
18. The circle of radius 4, centered at $(1, -3)$, oriented counterclockwise.
19. (a) Use a graphing utility to generate the trajectory of a particle whose equations of motion over the time interval $0 \le t \le 5$ are
$$x = 6t - \tfrac{1}{2}t^3, \quad y = 1 + \tfrac{1}{2}t^2$$

(b) Make a table of x- and y-coordinates of the particle at times $t = 0, 1, 2, 3, 4, 5$.
(c) At what times is the particle on the y-axis?
(d) During what time interval is $y < 5$?
(e) At what time does the x-coordinate of the particle reach a maximum?

20. (a) Use a graphing utility to generate the trajectory of a paper airplane whose equations of motion for $t \geq 0$ are

$$x = t - 2\sin t, \quad y = 3 - 2\cos t$$

(b) Assuming that the plane flies in a room in which the floor is at $y = 0$, explain why the plane will not crash into the floor. [For simplicity, ignore the physical size of the plane by treating it as a particle.]
(c) How high must the ceiling be to ensure that the plane does not touch or crash into it?

21–22 Graph the equation using a graphing utility.

21. (a) $x = y^2 + 2y + 1$
(b) $x = \sin y, \ -2\pi \leq y \leq 2\pi$

22. (a) $x = y + 2y^3 - y^5$
(b) $x = \tan y, \ -\pi/2 < y < \pi/2$

23. (a) By eliminating the parameter, show that the equations

$$x = x_0 + (x_1 - x_0)t, \quad y = y_0 + (y_1 - y_0)t$$

represent the line passing through the points (x_0, y_0) and (x_1, y_1).
(b) Show that if $0 \leq t \leq 1$, then the equations in part (a) represent the line segment joining (x_0, y_0) and (x_1, y_1), oriented in the direction from (x_0, y_0) to (x_1, y_1).
(c) Use the result in part (b) to find parametric equations for the line segment joining the points $(1, -2)$ and $(2, 4)$, oriented in the direction from $(1, -2)$ to $(2, 4)$.
(d) Use the result in part (b) to find parametric equations for the line segment in part (c), but oriented in the direction from $(2, 4)$ to $(1, -2)$.

24. Use the result in Exercise 23 to find
(a) parametric equations for the line segment joining the points $(-3, -4)$ and $(-5, 1)$, oriented from $(-3, -4)$ to $(-5, 1)$
(b) parametric equations for the line segment traced from $(0, b)$ to $(a, 0)$, oriented from $(0, b)$ to $(a, 0)$.

25. (a) Suppose that the line segment from the point $P(x_0, y_0)$ to $Q(x_1, y_1)$ is represented parametrically by

$$\begin{aligned} x &= x_0 + (x_1 - x_0)t, \\ y &= y_0 + (y_1 - y_0)t \end{aligned} \qquad (0 \leq t \leq 1)$$

and that $R(x, y)$ is the point on the line segment corresponding to a specified value of t (see the accompanying figure). Show that $t = r/q$, where r is the distance from P to R and q is the distance from P to Q.
(b) What value of t produces the midpoint between points P and Q?
(c) What value of t produces the point that is three-fourths of the way from P to Q?

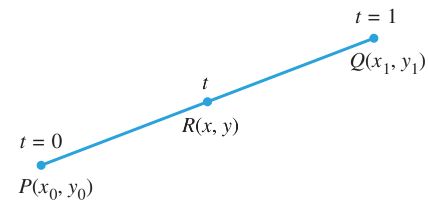


Figure Ex-25

26. Find parametric equations for the line segment joining $P(2, -1)$ and $Q(3, 1)$, and use the result in Exercise 25 to find
(a) the midpoint between P and Q
(b) the point that is one-fourth of the way from P to Q
(c) the point that is three-fourths of the way from P to Q.

FOCUS ON CONCEPTS

27. In each part, match the parametric equation with one of the curves labeled (I)–(VI), and explain your reasoning.
(a) $x = \sqrt{t}, \ y = \sin 3t$ (b) $x = 2\cos t, \ y = 3\sin t$
(c) $x = t\cos t, \ y = t\sin t$
(d) $x = \dfrac{3t}{1 + t^3}, \ y = \dfrac{3t^2}{1 + t^3}$
(e) $x = \dfrac{t^3}{1 + t^2}, \ y = \dfrac{2t^2}{1 + t^2}$
(f) $x = \frac{1}{2}\cos t, \ y = \sin 2t$

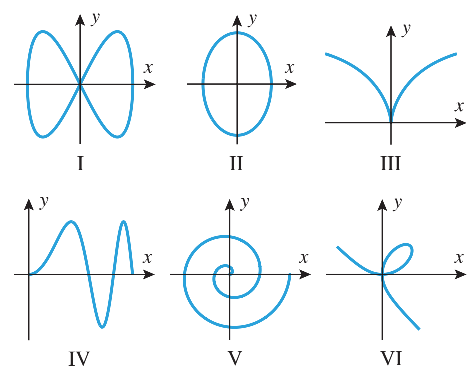


Figure Ex-27

28. Use a graphing utility to generate the curves in Exercise 27, and in each case identify the orientation.

29. Explain why the parametric curve

$$x = t^2, \quad y = t^4 \quad (-1 \leq t \leq 1)$$

does not have a definite orientation.

30. (a) In parts (a) and (b) of Exercise 23 we obtained parametric equations for a line segment in which the parameter varied from $t = 0$ to $t = 1$. Sometimes it is desirable to have parametric equations for a line segment in which the parameter varies over some other interval, say $t_0 \leq t \leq t_1$. Use the ideas in Exercise 23 to show that the line segment joining the points (x_0, y_0) and (x_1, y_1) can be represented parametrically as

$$x = x_0 + (x_1 - x_0)\frac{t - t_0}{t_1 - t_0}, \qquad (t_0 \le t \le t_1)$$
$$y = y_0 + (y_1 - y_0)\frac{t - t_0}{t_1 - t_0}$$

(b) Which way is the line segment oriented?
(c) Find parametric equations for the line segment traced from $(3, -1)$ to $(1, 4)$ as t varies from 1 to 2, and check your result with a graphing utility.

31. (a) By eliminating the parameter, show that if a and c are not both zero, then the graph of the parametric equations

$$x = at + b, \quad y = ct + d \qquad (t_0 \le t \le t_1)$$

is a line segment.
(b) Sketch the parametric curve

$$x = 2t - 1, \quad y = t + 1 \qquad (1 \le t \le 2)$$

and indicate its orientation.

32. (a) What can you say about the line in Exercise 31 if a or c (but not both) is zero?
(b) What do the equations represent if a and c are both zero?

33–36 Use a graphing utility and parametric equations to display the graphs of f and f^{-1} on the same screen.

33. $f(x) = x^3 + 0.2x - 1, \quad -1 \le x \le 2$

34. $f(x) = \sqrt{x^2 + 2} + x, \quad -5 \le x \le 5$

35. $f(x) = \cos(\cos 0.5x), \quad 0 \le x \le 3$

36. $f(x) = x + \sin x, \quad 0 \le x \le 6$

37. Parametric curves can be defined piecewise by using different formulas for different values of the parameter. Sketch the curve that is represented piecewise by the parametric equations

$$x = 2t, \quad y = 4t^2 \qquad \left(0 \le t \le \tfrac{1}{2}\right)$$
$$x = 2 - 2t, \quad y = 2t \qquad \left(\tfrac{1}{2} \le t \le 1\right)$$

38. Find parametric equations for the rectangle in the accompanying figure, assuming that the rectangle is traced counterclockwise as t varies from 0 to 1, starting at $\left(\frac{1}{2}, \frac{1}{2}\right)$ when $t = 0$. [*Hint:* Represent the rectangle piecewise, letting t vary from 0 to $\frac{1}{4}$ for the first edge, from $\frac{1}{4}$ to $\frac{1}{2}$ for the second edge, and so forth.]

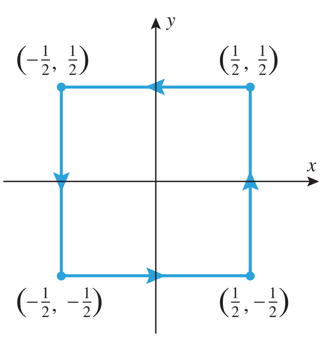


Figure Ex-38

39. (a) Find parametric equations for the ellipse that is centered at the origin and has intercepts $(4, 0)$, $(-4, 0)$, $(0, 3)$, and $(0, -3)$.
(b) Find parametric equations for the ellipse that results by translating the ellipse in part (a) so that its center is at $(-1, 2)$.
(c) Confirm your results in parts (a) and (b) using a graphing utility.

40. We will show later in the text that if a projectile is fired from ground level with an initial speed of v_0 meters per second at an angle α with the horizontal, and if air resistance is neglected, then its position after t seconds, relative to the coordinate system in the accompanying figure is

$$x = (v_0 \cos\alpha)t, \quad y = (v_0 \sin\alpha)t - \tfrac{1}{2}gt^2$$

where $g \approx 9.8$ m/s^2.
(a) By eliminating the parameter, show that the trajectory is a parabola.
(b) Sketch the trajectory if $\alpha = 30°$ and $v_0 = 1000$ m/s.

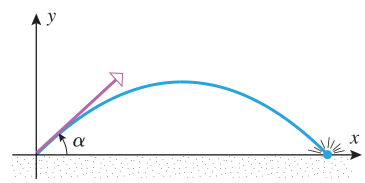


Figure Ex-40

41. A shell is fired from a cannon at an angle of $\alpha = 45°$ with an initial speed of $v_0 = 800$ m/s.
(a) Find parametric equations for the shell's trajectory relative to the coordinate system in Figure Ex-40.
(b) How high does the shell rise?
(c) How far does the shell travel horizontally?

42. A robot arm, designed to buff flat surfaces on an automobile, consists of two attached rods, one that moves back and forth horizontally, and a second, with the buffing pad at the end, that moves up and down (see Figure Ex-42 on the next page).
(a) Suppose that the horizontal arm of the robot moves so that the x-coordinate of the buffer's center at time t is $x = 25 \sin \pi t$ and the vertical arm moves so that the y-coordinate of the buffer's center at time t is $y = 12.5 \sin \pi t$. Graph the trajectory of the center of the buffing pad.
(b) Suppose that the x- and y-coordinates in part (a) are $x = 25 \sin \pi a t$ and $y = 12.5 \sin \pi b t$, where the constants a and b can be controlled by programming the robot arm. Graph the trajectory of the center of the pad if $a = 4$ and $b = 5$.
(c) Investigate the trajectories that result in part (b) for various choices of a and b.

43. Describe the family of curves described by the parametric equations

$$x = a\cos t + h, \quad y = b \sin t + k \qquad (0 \le t \le 2\pi)$$

if
(a) h and k are fixed but a and b can vary

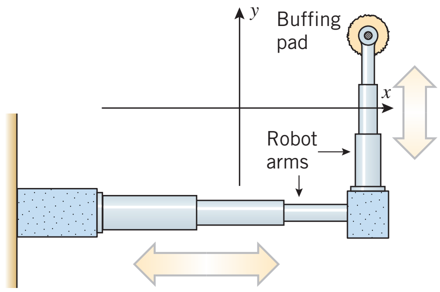


Figure Ex-42

(b) a and b are fixed but h and k can vary
(c) $a = 1$ and $b = 1$, but h and k vary so that $h = k + 1$.

44. A ***hypocycloid*** is a curve traced by a point P on the circumference of a circle that rolls inside a larger fixed circle. Suppose that the fixed circle has radius a, the rolling circle has radius b, and the fixed circle is centered at the origin. Let ϕ be the angle shown in the accompanying figure, and assume that the point P is at $(a, 0)$ when $\phi = 0$. Show that the hypocycloid generated is given by the parametric equations

$$x = (a - b)\cos\phi + b\cos\left(\frac{a-b}{b}\phi\right)$$

$$y = (a - b)\sin\phi - b\sin\left(\frac{a-b}{b}\phi\right)$$

45. If $b = \frac{1}{4}a$ in Exercise 44, then the resulting curve is called a four-cusped hypocycloid.
(a) Sketch this curve.

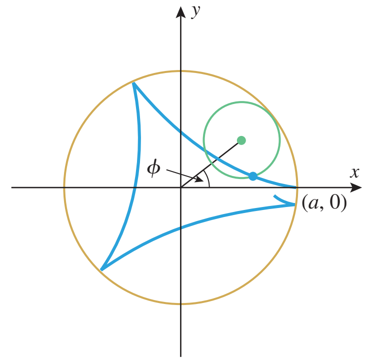


Figure Ex-44

(b) Show that the curve is given by the parametric equations

$$x = a\cos^3\phi, \quad y = a\sin^3\phi$$

(c) Show that the curve is given by the equation

$$x^{2/3} + y^{2/3} = a^{2/3}$$

in rectangular coordinates.

46. (a) Use a graphing utility to study how the curves in the family

$$x = 2a\cos^2 t, \quad y = 2a\cos t\sin t \qquad (-2\pi < t < 2\pi)$$

change as a varies from 0 to 5.
(b) Confirm your conclusion algebraically.
(c) Write a brief paragraph that describes your findings.

✔ QUICK CHECK ANSWERS 1.8

1. $x = 3 + 2\cos t$, $y = 5 + 2\sin t$ $(0 \le t \le 2\pi)$ **2.** $x = a\cos t$, $y = b\sin t$ $(0 \le t \le 2\pi)$ **3.** $\frac{3}{4}$; 2.75
4. $x = f(1 - t)$, $y = g(1 - t)$ **5.** $f(t)$

CHAPTER REVIEW EXERCISES

Graphing Utility

1. Sketch the graph of the function

$$f(x) = \begin{cases} -1, & x \le -5 \\ \sqrt{25 - x^2}, & -5 < x < 5 \\ x - 5, & x \ge 5 \end{cases}$$

2. Use the graphs of the functions f and g in the accompanying figure to solve the following problems.
(a) Find the values of $f(-2)$ and $g(3)$.
(b) For what values of x is $f(x) = g(x)$?
(c) For what values of x is $f(x) < 2$?
(d) What are the domain and range of f?
(e) What are the domain and range of g?
(f) Find the zeros of f and g.

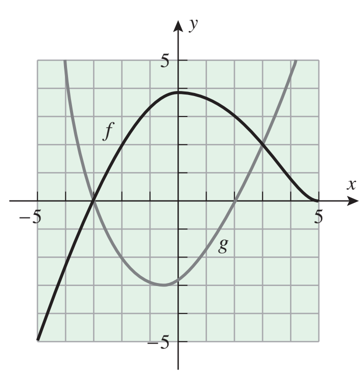


Figure Ex-2

3. A glass filled with water that has a temperature of 40°F is placed in a room in which the temperature is a constant